Rice Formula:

Extensions and Applications.

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Resumen

El hilo conductor de esta monografía es la Fórmula de Rice para el número de cruces de un proceso estocástico con un nivel o altura dada.

Trabajamos sobre dos tipos de problemas vinculados con dicha fórmula (familia de fórmulas).

Por un lado, en la primer parte de la tesis, abordamos el problema de extender la Fórmula de Rice a procesos cuyas trayectorias tengan saltos, se obtiene tal fórmula para un proceso que es la suma de dos procesos independientes: un proceso de trayectorias regulares (suaves) al cual se le pueda aplicar la versión tradicional y uno de saltos.

Se dan expresiones integrales para el número medio de cruces continuos y para el número medio de cruces discontinuos (saltos) al nivel dado. Para ello es necesario aplicar técnicas distintas, unas de procesos continuos y otras de procesos puntuales.

Luego, se presenta un par de ejemplos de cálculo concreto de estas fórmulas y se compara qué tipo de cruces predomina a medida que el nivel tiende a infinito. Además, en uno de estos ejemplos y en otro de un proceso puramente de saltos, se estudia la cola de la distribución del máximo cuando el nivel crece a infinito.

Por otro lado, en la segunda parte de la tesis, nos dedicamos a la aplicación de la Fórmula de Rice (tradicional, es decir, para procesos suaves) para estudiar el número de raíces de polinomios aleatorios y sistemas de polinomios aleatorios.

Más concretamente, en primer lugar, abordamos los Polinomios Aleatorios Trigonométricos Clásicos definidos como combinaciones lineales de cosenos con coeficientes independientes Gaussianos. Se obtiene la varianza asintótica y un Teorema Central del Límite para el número de ceros de este tipo de polinomios. En este punto, juega un rol protagónico el llamado Caos de Wiener.

Finalmente, estudiamos sistemas de ecuaciones polinomiales aleatorios complejos, para ello adaptamos la Fórmula de Rice sobre variedades a este contexto. Luego, usamos estas herramientas para dar un posible camino de prueba del Teorema de Bézout sobre el número de soluciones de tales sistemas. Obtenemos la prueba en algunos casos particulares, entre ellos el Teorema Fundamental del Álgebra y los sistemas cuadrados cuadráticos (grado dos) de cualquier orden.

Abstract

The main line of this monograph is Rice Formula for the number of crossings through a fixed level by a stochastic process.

We are concerned with two types of problems associated with this formula (family of formulas):

On one hand: in the first part of the thesis, we focus on the problem of extending Rice's formula to processes which trajectories include jumps, we obtain such formula for processes that can be written as the sum of two independent processes, one with smooth paths and a pure jump one.

We give integral expressions for the mean number of continuous and discontinuous crossings through the given level. In order to do that, we need to use different tools, from continuous processes theory to point processes theory.

Afterwards, departing from these results we present two examples of actual computation of these formulas and we compare which kind of crossing predominate as the level tends to infinity. Besides, in one of these examples and in another example of a pure jump process, we study the tail of the distribution of the maximum as the level grows to infinity.

On the other hand, in the second part of the thesis, we are concerned with the application of (classical, that is, for smooth processes) Rice Formula to study the number of roots of random polynomials and to systems of random polynomials.

More precisely, in the first place, we study Random Classical Trigonometric Polynomials, which are defined as linear combinations of cosines with independent Gaussian coefficients. We obtain the asymptotic variance and a Central limit theorem for the number of roots on an interval. At this point, a key role is played by the so called Wiener Chaos.

Finally, we study systems of random polynomials complex equations. For that, we adapt Rice formula for manifolds to that context. Afterwards, we use these tools in order to give a possible approach to the proof of Bézout Theorem about the number of roots of such systems. We do obtain the proof in some particular cases, for example, The Fundamental Theorem of Algebra and for square quadratic (degree 2) systems of equations of any order.

a la memoria de Mario Wschebor

Agradecimientos

Esta monografía es el fruto de cinco años de trabajo. Su contenido refleja marchas y contra marchas. en este sentido, quedaron por el camino varios problemas, incluyendo el original y que motivó el planteo inicial de la tesis, sobre conectividad algebraica de grafos aleatorios. En cambio, en la versión final aparecen problemas que al inicio no sospechábamos.

Esta monografía fue posible gracias a la ayuda y el apoyo de muchas personas e instituciones a las que quiero agradecer.

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Je veux remercier aussi à Jean-Marc Azaïs par me accueillir à Toulouse plusieurs fois où nous avons parlé sur les problèmes de la thèse, en particulier de la géométrie associé à la Formule de Rice en dimension plus grand que un. Aussi par partager “el verdadero Tannat”.

También, durante este tiempo, tuve el gran gusto de compartir trabajo con Diego Armentano, con quien me divertí mucho y aprendí más. El Capítulo 7 da nuestra respuesta parcial a un problema planteado en 2011 por Mario Wschebor.

Durante este tiempo me radiqué en Salto, compartí el proyecto y la creación del Departamento de Matemática y Estadística del Litoral (DMEL), primero de Matemática en el interior del país. Agradezco a mis compañeros del DMEL, y también del Departamento del Agua y del Laboratorio de Virología por compartir esta experiencia.

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En último lugar en la enumeración pero primero en importancia quiero agradecer a Hernán, a Vicente y a Elisa por su infinita paciencia y apoyo.

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Chapter 1

Introduction

The main stream of this thesis is Rice Formula, also known as Kac-Rice Formula, on the mean number of crossings through a fixed level by a stochastic process on a time interval.

The basic underlying problem for Rice Formula is the following: given a stochastic process defined on an interval, to study the random variable *number of points* where the process crosses through a fixed level u.

This problem is intimately related with the barrier problem for the same process, that is, with the problem of studying the probability that the process exceeds certain level on the interval. The connection between these problems have been used to obtain bounds for the tail of the distribution of the maximum of the process on the interval in terms of Rice Formula. These are classical problems in Probability Theory.

The multivariate version of the level crossing counting problem is as follows: given a stochastic field defined on a parameter space $\mathcal{T}$, usually (a subset of) $\mathbb{R}^d$, and a level u in $\mathbb{R}^{d'}$, $d' \leq d$, respectively, to study the random variable *number of points* in $\mathcal{T}$ where the process takes the level u in the case $d = d'$; or the random variable *geometric measure* of the level set for level u when $d' < d$.

These problems appear in diverse topics of science and technology. For instance, there are many situations where there are critical levels, above (or below) which extreme events occur. For example, if the height of sea waves exceeds certain level, they can cause damages on oil platforms or on ships; or if the concentration of certain substances may be harmful to health if too high; and so on.

To be more precise, we can enumerate a few examples, in different fields and areas, where the methodology of Rice Formulas has been applied: telecommunications and signal processing by Rice [73, 74]; reliability theory in engineering by Rychlik [75]; oceanography: the height of sea waves by Longuett & Higgins

[59] and by Azaïs, León & Wschebor [10]; physics and astronomy: random mechanics by Krée [51], the Shot Noise process by Biermé & Desolneux [18] and Microlensing by Petters, Rider & Teguia [69]; algorithm complexity for solving random systems of polynomial equations by Cucker, Crick, Malajovic & Wschebor [24] and by Armentano & Wschebor [8], etc.

On this context, on the present monograph we deal with two problems:

1. to extend Rice Formula to new classes of processes. More precisely, we consider processes whose paths are càdlàg and piecewise smooth but may include jumps with finite intensity. We are interested in studying both, the continuous and discontinuous crossings of such processes through a fixed level.

   The first part of the thesis is concerned with this problem and some related topics.

2. to extend the applications of classical Rice Formula to polynomial equations, and systems of polynomial equations. We consider two ensembles of random polynomials: classical trigonometric polynomials and study their asymptotic behavior as the number of terms grows to infinity; complex systems of m variables and m equations with Weyl randomization and Bezout's theorem about its mean number of roots.

   The second part of the thesis is concerned with these applications.

In the Preliminaries we review with some care the development of this method and describe our contributions with more details.

This monograph is associated with the papers: Rice Formulas for processes with jumps and applications, Dalmao & Mordecki [27], and CLT for Classical Trigonometric Polynomials, Dalmao & León [26]. Besides, the work on complex systems of equations in Chapter 6 is in collaboration with Diego Armentano, see his PhD thesis [7], it is worth to say that the case of quadratic systems of equations does not appear there.

We end this introduction with some general remarks about Rice Formula.

To fix ideas, we must say that by a Rice Formula we understand an expression (generally an integral formula) for the mathematical expectation, or for a higher moment, of the random variable *number of crossings* by a stochastic process through a fixed level in terms of the parameters and local properties of the underlying process.

Generally speaking, very few is known about the distribution of the number of crossings and of the maximum of the process on a finite interval, excepting

some very particular cases. Though for the case of processes of irregular paths, as martingales or Lévy processes, there are well known results, for processes with smooths paths, or piecewise smooth, not very much is known. For the former class of processes see for instance Ikeda & Watanabe [41] or Karatzas & Shreeve [46], for the latter class see Borovkov & Last [20, 21]. There are also results relating the crossings (or level sets) of processes with smooth and processes with non-smooth paths, by regularizing the paths by convolution with some convenient kernel, see Wschebor [81] for a review. See also the Introduction of the book Azaïs & Wschebor [12] for a review of known results on the distribution of the maximum.

Thus, Rice Formula is a very relevant tool, as it gives some information about this distribution, not directly but through its moments.

One of the main characteristics of the formula is that it expresses a global property such as the number of crossings, which depend on the whole parameter space, in terms of the parameters and local properties of the process. On the other hand, one of the drawbacks is that the resulting expression may be very hard or impossible to compute explicitly. But, it does can be computed, or at least bounded, on some well chosen cases.

This family of formulas have a very rich history, we will talk about it briefly in the Preliminaries, see page 5. Now, we limit ourselves to analyze the original formula due to Rice in 1944. It was established for a stationary, centered Gaussian process with covariance function r on the interval $[0, T]$, and states that the expectation of the number of up-crossings, U_u, through the level u by the process is given by

$$\mathbb{E}\, U_u = T\frac{\sqrt{\lambda_2}}{2\pi}e^{-u^2/2} = T\sqrt{\frac{\lambda_2}{2\pi}}\varphi(u), \tag{1.1}$$

where the process has been normalized in order to have $r(0) = 1$. As usual, φ is the standard Gaussian density function on $\mathbb{R}$ and λ_2 is the second spectral moment of the process.

Observe that from Equation (1.1), it follows that the maximum intensity of crossings is reached at the level $u = 0$ (generally, if the process is not centered, the maximum intensity of crossings is reached when u equals the mean of the process) and that this intensity decays exponentially fast as the level u moves away from zero.

Besides, it is well known from Harmonic Analysis that the second spectral moment λ_2 gives a measure of the local oscillation of the paths of $\mathcal{X}$, that is, the larger is the value of λ_2 the more oscillations the paths present, then, it is natural that for processes with large value of λ_2 there be more crossings (if there is a crossing, the oscillations tend to produce other crossings near the first one).

It is also important to remark that Equation (1.1) implies that the mean number of crossings is proportional to the one dimensional density function of the stationary process. This is one of the main features of Rice Formula and it will appear several times in the sequel.

Undoubtedly, the most studied case in the literature is that of Gaussian processes, specially when they are stationary. The main reason for such preference is quite practical, since in that case there is a great simplification of the ingredients in the formulas, with respect to their definition and actual computation. For instance, the regularity of the paths implies the regularity of the moments and densities and vice versa, or specially the remarkable and peculiar fact that on the Gaussian case conditional expectations can be computed easily by means of the procedure of Gaussian regression. Furthermore, in many situations, the Gaussian law, with convenient parameters, is invariant under the orthogonal (unitary) group on the parameter space, that is, under isometries, this fact permits to remove the integrals and to compute everything at a single and convenient point. See the Appendix on Gaussian processes for some details on Gaussian distribution, page 131.

Apart from the Gaussian case, few is known, since the available results either impose very restrictive conditions to the processes or the processes are assumed to have a very particular form and the prove of the formula is based on ad hoc arguments, see for instance Azaïs & Wschebor [12], Biermé & Desolneux [18] and Alodat & Aludaat [4].

Chapter 2

Preliminaries - Rice Formula

On this chapter we give a general panorama of the state of the art in what respect to Rice Formulas and present with some detail the contributions of the present monograph. For simplicity, we deal mainly with the univariate case.

2.1 Introduction

Consider a (real valued) stochastic process $\mathcal{X} = (X(t) : t \in [0,T])$ with smooth paths, that is, the functions $X(\omega, \cdot)$ are of class C^k on $[0,T]$, for some $k \geq 1$, with probability one.

Eventually, we will allow the paths to have finitely many jumps.

As mentioned in the Introduction, the problem of studying the number of level crossings of the process $\mathcal{X}$ and the problem of studying its extreme values are classical in Probability Theory.

Let us start with the definition of a crossing. We avoid pathologies, see Leadbetter, Lindgren & Rootzén [54, Section 7.2] for a wider discussion of the definition.

Definition 2.1. *Let $u \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be a càdlàg function with isolated jumps and of class C^1 between jumps, we say that:*

- *f has a continuous crossing through the level u at $s \in (0,T)$ if f is continuous at s, $f(s) = u$ and $f'(s) \neq 0$.*
- *f has a discontinuous crossing through the level u at $s \in (0,T)$ if $(f(s^-) - u)(f(s) - u) < 0$. As usual, $f(s^-) = \lim_{t \to u, t<u} f(t)$.*

If $f'(s) > 0$ in the continuous case, or $f(s^-) < u < f(s)$ in the discontinuous

one, we say that f has an up-crossing at s, otherwise the crossing is said to be a down-crossing.

Remark 2.1. *The paths of the processes we deal with have isolated zeros.*

We denote the set of crossings through the level u by the process $\mathcal{X}$ on the interval $[0,T]$ by $C_u = C_u(\mathcal{X},[0,T])$, the number of these crossings by $N_u = \#C_u$, that is

$$C_u = \{s \in (0,T) : f \text{ has a crossing at } s\}, \quad N_u = \#C_u.$$

Analogously the number of up-crossing is denoted by U_u and the number of down-crossings by D_u. To distinguish between continuous and discontinuous crossings we use the super indices c, d respectively.

There are some classical results which give general conditions for a process in order not to have critical points at a given level. They permit to identify, in the continuous crossings case, the roots of the equation $X(t) = u$ with the crossings of that level. See Azaïs & Wschebor [12] and Kratz [48] and references therein.

Theorem 2.1 (Bullinskaia)**.** *Let $\mathcal{X}$ be a stochastic process with paths of class C^1 defined on an interval I in $\mathbb{R}$. If for each $t \in I$, the random variable $X(t)$ has a density, and the density is bounded for $t \in I$ and x on a neighborhood of u, then, X has no critical points at the level u with probability one.*

Theorem 2.2 (Ylvisaker)**.** *Let $\mathcal{Z}$ be a Gaussian process with continuous paths over a separable compact topological space $\mathcal{T}$ with positive variance for all $t \in \mathcal{T}$, then, $\mathcal{Z}$ has no critical points at the level u with probability one.*

Theorem 2.3 (Tsirelson)**.** *Let $\mathcal{Z}$ be a Gaussian process over an arbitrary parameter space $\mathcal{T}$, then, for each u such that*

$$\mathbb{P}\left(\sup_{t\in\mathcal{T}} X(t) < u\right) > 0,$$

almost surely, there are not critical points at level u.

Classical Rice formula, that is, Rice Formula for processes with path of class at least C^1, is based on the Kac Counting Formula for the continuous crossings through a fixed level by a real function on an interval I, $[0,T]$ on our case:

$$N_u(f,[0,T]) = \lim_{\delta\to 0} \frac{1}{2\delta} \int_{[0,T]} |f'(t)|\, \mathbb{I}\{|f(t)-u| < \delta\} dt,$$

where $\mathbb{I}A$ stands for the indicator, or characteristic, function of the set A. See Figure 2.1.

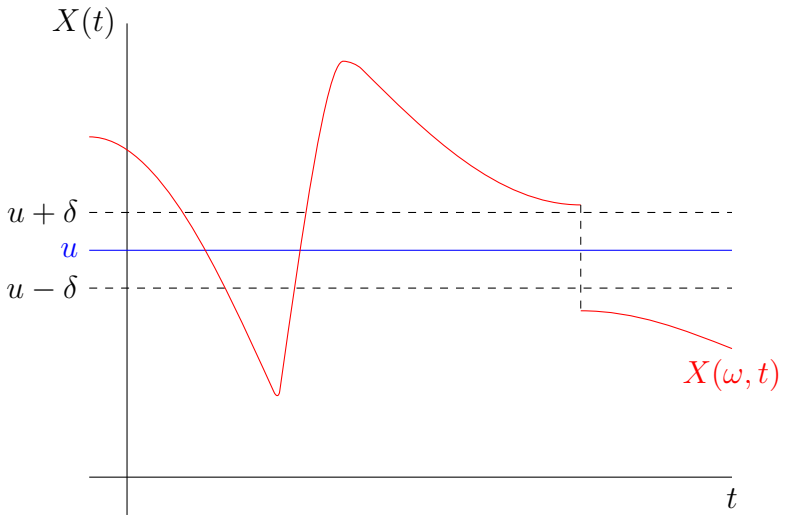

Figure 2.1: Kac Counting Formula

Roughly speaking, in order to obtain Rice Formula departing from Kac Counting Formula, we apply the latter to the paths of the process $\mathcal{X}$, i.e: to each function $f(\cdot) = X(\cdot, \omega)$ and take expectations on both sides.

Informally, we can think on Kac Counting Formula in terms of the Dirac delta measure. In effect, assume that the (continuous) crossings through the level u are isolated, hence, in a small enough neighborhood I_i of one of the crossings s_i, we may apply the change of variable formula to obtain

$$1 = \int_{\mathbb{R}} \delta_u(y)dy = \int_{I_i} \delta_u(f(t))|f'(t)|dt.$$

Summing over the crossings $s_i : i = 1, \ldots, N_u$ we obtain

$$N_u = \int_{[0,T]} \delta_u(f(t))|f'(t)|dt.$$

See Adler & Taylor [1, page 265] and Kratz & León [49].

On this manner, we see that the limit when δ tends to zero in the above formula represents nothing but an approximation to the unity.

Of course, one can use other approximations of the unity, for instance, Kratz & León [49] and Azaïs & León [9] use Gaussian kernels.

In the case $u = 0$, in 1943 using the Fourier transform of the Dirac Delta function, Kac [45] expressed the number of zeros, N_0, of a C^1 function as

$$N_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^t \cos(\zeta f(s))|f'(s)|dsd\zeta.$$

A generalization for other levels is given in Kratz & León [49].

It is a key fact that Kac Counting Formula ignores, that is, it does not take into account, the discontinuous crossings through level u, in effect, for δ small enough the strip $(u-\delta, u+\delta)$ is completely included on the jump $(f(t^-), f(t))$ and therefore the indicator function vanishes, see Figure 2.1 again.

2.2 Brief summary of the history and methods

For the sake of readability and of better understanding of the contributions of the monograph, we include in the following lines a very brief summary of the existing results related to Rice Formula.

In particular, we present different approaches and classes of processes for which some of the versions of Rice Formula hold true.

2.2.1 The original Formula

The original Rice Formula obtained by Rice in 1944, 1945 in [73, 74] for a stationary centered Gaussian process with variance one on the interval $[0,1]$ states that:

$$\mathbb{E}\, U_u = \frac{\sqrt{\lambda_2}}{2\pi} e^{-u^2/2}, \tag{2.1}$$

where λ_2 stands for the second spectral moment of the process at $t=0$, that is, λ_2 measures the local oscillation of the paths of X, see Adler & Taylor [1]

This result was obtained by Rice by informal arguments in his work about telecommunications and signal processing [73, 74], see also Rainal [71]. Afterwards, it was formally proved under successively weaker conditions by Ivanov (1960), Bulinskaya (1961), Itô (1964) and Ylvisaker (1965). The final result is that Formula (2.1) holds true (with either finite or infinite value of λ_2) whenever the process has almost surely continuous paths, see Leadbetter & Spaniolo [55] and Azaïs & Wschebor [12] and references therein.

One of the most surprising consequences of Formula (2.1) is that the mean number of crossings is proportional to the density of the process at the initial time (or at an arbitrary time instant since the process is stationary), see Leadbetter & Spaniolo [55].

2.2.2 A general formula for stationary processes

At this point we follow Leadbetter and Spaniolo [55], the following general formula holds true for stationary processes

$$\mathbb{E}\, U_u = \int_0^\infty z p(u, z) dz \tag{2.2}$$

where p is the joint density of the stationary process X and its derivative X' at (any time) t, here the derivative is understood in some convenient way. Further, it is easy to extend this formula to non-stationary processes adding an integral w.r.t. to t.

In that sense, different classes of derivatives can be used: Marcus [61] consider the derivative in the sense of absolute continuity of X; Albin [2] consider the derivative in the L^1 sense (subject to regularity conditions). Following Azaïs & Wschebor [12], we interpret the derivative path wise.

Besides, from Equation (2.2) follows the formula for the mean number of crossings in terms of the conditional expectation for the stationary case.

Following Ylvisaker (1965) and Leadbetter (1966), Leadbetter and Spaniolo use polygonal processes in order to approximate the original process with simpler ones. It is clear that the main advantage of polygonal paths is that each segment of the polygonal path can cross only once the given level.

The stationarity of the process is a key fact to enable or, at least, to simplify the computations in order to obtain this formula. For instance, if we take the polygonal approximation method, by dividing the original interval in a large number of sub-intervals of the same length, the law of the process is the same at each one of them and the sub-intervals are more suited to treat the local properties.

More precisely, they set $q > 0$, and approximate the number of up-crossings through the level u by the process X, U_u, by the number of points of the form jq in $[0, 1]$ such that $X(jq) < u < X((j + 1)q)$. Therefore

$$\mathbb{E}\, U_u \approx J_q(u) = q^{-1}\mathbb{P}\{X(0) < u < X(q)\} = \mathbb{P}\{u - qZ_q < X(0) < u\}$$

being $Z_q = (X(q) - X(0))/q$. Then, the authors state, [55, Th.1], that for stationary processes with continuous distribution, $\mathbb{E}\, U_u = \lim_{q\to 0} J_q(u)$, using the second form given for J to avoid the degeneracy of the distribution, and the limit always exist either finite or infinite.

This is the basic result, under differentiability assumptions Formula (2.2) and other related formulas follow.

In particular, Theorem 2 of [55] state that if $X(0)$ and $X(q)$ have joint density $f_q(x,z)$, then $X(0)$ and Z_q have joint density $g_q(x,z) = qf_q(x, x+qz)$ and

$$\mathbb{E}\, U_u = \lim_{q\to 0} \int_0^\infty dz \int_0^z g_q(u-qx, z)dx,$$

Further, this limit always exists, but it can be infinite. If in addition $g_q(u - qx, z) \to_q p(u,z)$, then, Formula (2.2) follows provided some domination is assumed. The final step is to interpret p as the joint density of the process and its derivative.

2.2.3 Almost sure formulas

Usually, when dealing with level crossings and Rice Formulas one needs that such a formula holds true for every level u, since, for instance if the interest relies on the critical points, the formula must be valid for the derivative of the process at the level $u = 0$.

But, in some circumstances it suffices to state an almost sure version with respect to Lebesgue measure, that is, a formula which holds true for u in a set of total Lebesgue measure rather than for all u in $\mathbb{R}$. The almost sure version can be obtained under much weaker and less restrictive conditions.

For instance, Rychlik [75] show that the formula

$$\mathbb{E}\, N_u = \mathbb{E}\,(|\dot{X}(0)|)p_{X(0)}(u)$$

holds almost surely whenever $\mathbb{E}\,|\dot{X}(0)|$ is finite and $X(0)$ has a density w.r.t. Lebesgue measure.

Examples are given in Rychlik [75] of situations where the almost sure version suffices, in all of them the author is ultimately interested in computing some intensity given in terms of the expectation of the number of crossings w.r.t. the level u or uses a Bayesian argument which also yields an expectation of N_u w.r.t. u.

The Bayesian argument is, roughly speaking, the following: in practical situations usually one does not know exactly the level u, perhaps due to the lack of precision in some measuring device, so it is quite reasonable to assign it some a priori probabilistic distribution (uniform in some small interval for example) and then to take the average on this distribution.

The almost sure approach was used by Brillinger [22] for the multidimensional case, by Geman and Horowitz [35] as a by product of the study of occupation

times, by Zähle [82] also for the multidimensional case but for the (Hausdorff) geometric measure of level sets and by Rychlik [75].

To end this discussion, let us mention that, in general, in order to obtain the formula for every level departing from the almost sure formula some kind of continuity of the number of crossings with respect to the level u is needed.

2.2.4 The general theory and some recent extensions

Our main reference on Rice Formulas is the book by Azaïs and Wschebor [12]. The authors present a general theory about Rice Formula. In particular, giving important extensions and applications of Rice Formula.

Formulas are given for the factorial moments, of order one and higher, of the number of crossings through a fixed level by a Gaussian or non-Gaussian stochastic process.

On the Gaussian case, the required conditions on the process are of regularity of the paths and of non-degeneracy of the finite-dimensional distributions of the process. On the general case (not necessarily Gaussian) strong conditions on the regularity of the density functions and conditional expectations are assumed. See the Appendix A, page. 131.

Besides, the authors provide formulas for random fields, in the cases when the domain has dimension larger than the co-domain. The remaining case is uninteresting. In the case of different dimensions the formulas involve the geometric (Hausdorff) measure of the level set. Furthermore, the authors provide formulas for counting the number of weighted crossings, for example, the weights may be the values of the derivatives at the crossings.

Furthermore, these formulas are extended to the case when the parameter space is a smooth manifold.

The book considers other aspects of the subject, as the Rice Series which give some well behaved numeric algorithms for approximating some functional of the number of level crossings; and some asymptotic for the number of crossings.

Finally, let us mention briefly the applications, they range from the distribution of the maximum of the process on an interval, to random polynomials, algorithmic complexity, gene detection, or the height of the sea waves (on an infinitely deep sea). It is worth to say that these authors have studied recently the problem of the tail of the maximum for a parameter space with a fractal structure, see Azaïs and Wschebor [13].

In what respect some new extensions, Biermé & Desolneux [18] obtain a Rice

Formula for the so called Shot Noise Process. More precisely, they assume that the process has Gaussian impulses at random epochs given by a Poisson process in $\mathbb{R}$; the Shot Noise process is stationary. The resulting formula is expressed in terms of the Fourier Transform (or characteristic function) of the process. The authors apply the formula for the stationary centered Gaussian process obtaining Classical Rice Formula and study some asymptotic in other cases. An interesting point in this work is a bound on the number of crossings of a sum of processes in terms of the number of crossings of their derivative processes. Finally, it is worth to say that this kind of processes is very important in Physics.

Alodat & Aludaat [4] obtain a Rice Formula for the Generalized Hyperbolic Process. This process is not Gaussian but it is stationary. As particular cases, the Generalized Hyperbolic Process includes chi-squared and Student processes.

2.2.5 Formula for processes with jumps

On the case in which the paths of the process are allowed to have jumps with finite intensity but are smooth between the jumps, very few is known. Basically, the only results available are due to Borovkov and Last [20, 21].

In 2007 Borovkov and Last [20], proved the Rice Formula for a class of processes called *Piecewise Deterministic Markov Processes*. These are stationary processes, they have jumps at random epochs with random magnitudes, but between its jumps the evolution of the process is deterministic, and described through a given function μ and its associated flux.

In particular, from this definition follows that given the level u, the continuous crossings through u only can be up-crossings on the case that $\mu(u) > 0$ and down-crossings on the case $\mu(u) < 0$, these facts simplify the structure of the process and strongly relate the continuous and discontinuous crossings.

Borovkov and Last's formula says that if $\mu(u) \neq 0$, then

$$\mathbb{E}\, N_u = |\mu(u)| p(u),$$

being p the density of the stationary distribution of the process.

Some interesting facts about the Piecewise Deterministic Markov Processes are that the jump part and the continuous part of the processes are not independent, that the one-dimensional distribution are (conditionally) degenerated and that the intensities of continuous and discontinuous crossings are linked.

It is worth to say that the major part of Borovkov and Last's article [20] is devoted to prove that as the interval increases to cover the positive axis and the level tends to infinity the point process of crossings through the level u, suitably

normalized, converge to a Poisson Process. This is the first result of this type for this kind of processes, but is a well known result for Gaussian processes.

2.3 Contributions

At this point, we are able to write down in some detail which are the contributions of the present monograph. These are twofold.

2.3.1 Processes with jumps

Part I of the thesis is dedicated to an extension of Rice Formula to a wider class of stochastic processes. This extension allow the process to have jumps with finite intensity, all the jumps epochs, the jumps magnitudes and the evolution between the jumps are (can be) stochastic.

We deal with both, continuous and discontinuous crossings.

Part I is organized as follows, in Chapter 3 we obtain formulas for the mean number of crossings, both continuous and discontinuous, through a fixed level by the process on a compact interval. On Chapter 4, we compute these formulas in some examples and apply the result to the asymptotic as the level goes to infinity and to the study of the tail of the distribution of the maximum on a compact interval.

The main new aspect in our formulas is that we study also the discontinuous crossings through the given level. Although, Borovkov and Last in [20] studied a special class of stochastic processes with jumps, there was a strong relation between continuous and discontinuous crossings.

See Dalmao & Mordecki [27].

2.3.2 Random Polynomials

Part II of the thesis is concerned with the application of Rice classical Formula to a pair of problems related to the number of roots of random polynomials and polynomial systems.

Part II is organized as follows, in Chapter 5 we consider the Classical Trigonometric ensemble of random polynomials which are very important in Physics. We follow Azaïs & León [9], who studied a stationary version of trigonometric polynomials. We prove a previous conjecture for the asymptotic variance of the number

of roots of such polynomials and a Central Limit Theorem for this ensemble. In the proof of these results, we use the Chaotic expansion.

See Dalmao & León [26].

On chapter 6, we consider random complex square systems of polynomial equations. We adapt Rice Formula to the complex framework and give some argument that should lead to a proof of Bézout's Theorem about the number of roots of such a system. We are able to prove some particular cases of this theorem, mainly the quadratic case of an $m \times m$ system whose equations have degree 2.

Part I

Processes with Jumps

As mentioned above, the first part of the thesis is dedicated to the extension of Rice Formula to a wider class of stochastic processes. More precisely, we want the paths of the process to have jumps with finite intensity and to count not only the continuous crossings but also the jumps (discontinuous) crossings through the level u.

This work began some time ago under the initiative of Mario Wschebor, and is largely inspired by his fundamental contribution in the field, masterly exposed by Azaïs and Wschebor in the book [12].

Part I is organized as follows.

In Chapter 3, we obtain integral formulas for the mean number of continuous and discontinuous (and total) level crossings of such processes on compact intervals.

In Chapter 4, we perform actual computations of these formulas in a few examples. Then, we apply these formulas for the study of the tail of the distribution of the maximum of a stationary process with Gaussian one-dimensional distribution and jumps and also for a pure jump process. Besides, we compare which kind of level crossings predominate as the level grows to infinity.

Chapter 3

Rice Formula for processes with jumps

In this chapter we present an extension of Rice Formula that allows the paths of the process to have jumps with finite intensity, that is, we consider a class of stochastic processes that may jump finitely many times on compact intervals and have a smooth (stochastic) evolution between these jumps (excepting Section 4). More precisely, we consider a process $\mathcal{X}$ which can be written in the form $\mathcal{X} = \mathcal{Z} + \mathcal{J}$ where $\mathcal{Z}$ is a process with continuously differentiable paths and $\mathcal{J}$ is a pure jump process, independent from $\mathcal{Z}$.

Thus, the process $\mathcal{Z}$ describes the continuous evolution of $\mathcal{X}$ and the process $\mathcal{J}$ describes the jumps (instants and magnitudes) of $\mathcal{X}$.

Observe that such a process can cross the level u in a continuous way but it also can cross u at one of the jump epochs. Our interest relies on both, continuous and discontinuous crossings through the level u. Up to our knowledge, the discontinuous crossings have not been considered in the literature.

We obtain formulas for the mean number of continuous and discontinuous crossings through a fixed level $u \in \mathbb{R}$ by such processes on a compact time interval In the next chapter, we apply these formulas to the study of the distribution function of the maximum of the process. We also include a generalization of Borovkov-Last's Rice-type formula to the non-stationary case

Naturally, when there are no jumps, our formulas reduce to the classical formulas.

We start with some preliminaries, including the basic definition and the description of the processes we deal with.

3.1 Preliminaries

We recall the definition of a crossing. Consider a càdlàg, C^1 between jumps, function $f : [0,T] \to \mathbb{R}$, $T > 0$. We say that f has a continuous crossing through the level u at $s \in (0,T)$ if f is continuous at s, $f(s) = u$ and $f'(s) \neq 0$; and a discontinuous crossing if $(f(s^-) - u)(f(s) - u) < 0$.

If $f'(s) > 0$ in the continuous case, or $f(s^-) < u < f(s)$ in the discontinuous one, we say that f has an up-crossing at s, otherwise the crossing is said to be a down-crossing.

Let $T > 0$ and $\mathcal{X} = (X(t) : t \in [0,T])$ be a stochastic process defined on the interval $[0,T]$. We assume that $\mathcal{X}$ can be written in the form $\mathcal{X} = \mathcal{Z} + \mathcal{J}$ where $\mathcal{Z}$ and $\mathcal{J}$ are independent processes on $[0,T]$ as described below.

The process $\mathcal{Z}$ describes the continuous evolution of the main process $\mathcal{X}$, we assume some regularity conditions on $\mathcal{Z}$, see Azaïs & Wschebor [12, Th. 3.4], namely:

*A*1 the paths of $\mathcal{Z}$ are C^1, almost surely.

*A*2 We assume that the density of $Z(t)$, $p_{Z(t)}(x)$, is jointly continuous for $t \in [0,T]$ and x in a neighborhood of u. Further, assume that for every $t, t' \in [0,T]$ the joint distribution of $(Z(t), Z'(t'))$ has a density $p_{Z(t),Z'(t')}(x,x')$ which is continuous w.r.t. t (t', x, x' fixed) and w.r.t. x at u (t, t', x' fixed).

*A*3 for every $t \in [0,T]$ there exists a continuous version of the conditional expectation

$$\mathbb{E}\left(Z'(t) | X(t) = x\right),$$

for x in a neighborhood of u.

*A*4 the modulus of continuity of Z' tends to 0 if $\delta \to 0$:

$$w(Z', \delta) := \sup_{0 \leq s < t \leq T,\, |t-s| < \delta} |Z'(t) - Z'(s)| \underset{\delta \to 0}{\to} 0.$$

For such a process, Rice Formula holds true, see again Azaïs & Wschebor [12].

Let us now describe the jump process $\mathcal{J}$.

The process $\mathcal{J} = (J_t : t \in [0,T])$ is based on a general point process $(\tau_n, \xi_n)_{n \in \mathbb{N}}$ on $[0,\infty) \times \mathbb{R}$, and is constructed via a family of Markov kernels $(P^{(n)}_{x,t})$, $(\pi^{(n)}_{x,t}) : n \in \mathbb{N}$ as follows, see Jacobsen [42]: set $\tau_0 = 0$ and draw ξ_0 according to the

initial distribution π_0, then, conditioned on the resulting value of ξ_0, say x_0, draw τ_1 with (conditional) distribution

$$\mathbb{P}\left(\tau_1 \in \cdot \mid \xi_0 = x_0\right) = P^{(1)}_{x_0}(\cdot).$$

Similarly, conditioned on the values of ξ_0, τ_1, say x_0, t_1 respectively, draw ξ_1 with distribution $\pi^{(1)}_{x_0,t_1}(\cdot)$. That is

$$\mathbb{P}\left(\xi_1 \in \cdot \mid \xi_0 = x_0, \tau_1 = t_1\right) = \pi^{(1)}_{x_0,t_1}(\cdot).$$

Then, conditioned on the preceding values and on $\xi_1 = x_1$ draw τ_2 with distribution $P^{(2)}_{(x_0,x_1),t_1}(\cdot)$ and so on.

Finally, for $\tau_n \leq t < \tau_{n+1}$ let $\nu_t = n$ and

$$J(t) = \sum_{k=0}^{\nu_t} \xi_k.$$

Therefore, τ_n represents the n-th jump instant (or epoch) and ξ_n represents the n-th jump magnitude (or increment) of the process $\mathcal{J}$. Besides, between the jump epochs $\mathcal{J}$ is constant.

Remark 3.1. *Equivalently, the kernels $(P^{(n)}_{x,t})$ can be used to obtain the actual value of the process at the jump instants, in that case we set $J(\tau_n) = \xi_n$.*

We can identify the (basic) marked point process $\big(\tau_n, J(\tau_n) : n \in \mathbb{N}\big)$ with its associated random counting measure (RCM) $\mu = \sum_{n=0}^{\nu_T} \delta_{(\tau_n, J(\tau_n))}$ on $[0,T] \times \mathbb{R}$, where $\delta_{(t,x)}$ is Dirac Delta measure concentrated at the point (t,x). Thus, the RCM μ of a Borel set C in $[0,T] \times \mathbb{R}$ is just the number of points (jump epochs and marks) of the marked point process that lie in C.

It can be proven, see Jacobsen [42, Chapter 4] or Appendix B in page 135, that there exists a random measure $L(dt, dy)$ on $[0,T] \times \mathbb{R}$, called the compensating measure of μ, such that the processes $t \mapsto L([0,t), A)$ are predictable for any $A \in \mathcal{B}$ and under quite general conditions (for example the absolute continuity of the kernels and the finiteness of the intensity of jumps), L can be written in terms of ordinary Lebesgue integrals. Furthermore

$$\mathbb{E} \int_{[0,T]\times\mathbb{R}} f d\mu = \mathbb{E} \int_{[0,T]\times\mathbb{R}} f L(dt, dy)$$

for any predictable function f.

Recall that, roughly speaking, to be predictable means that the value at t can be "predicted" from the previous values on $[0,t)$.

3.2 Rice Formula

We turn now to the extension of Rice Formula for the process $\mathcal{X}$ just introduced.

By the preceding assumptions on the process $\mathcal{X}$, we can apply the definition of a crossing to almost all of its paths. Denote N_u^c (resp. N_u^d) the (random) number of continuous (resp. discontinuous) crossings through the level u by the process $\mathcal{X}$ on the interval $[0,T]$ and by N_u the total number of crossings. Analogously U_u, U_u^c, U_u^d (D_u, D_u^c, D_u^d) denote respectively the total number, continuous, and discontinuous up-crossings (down-crossings).

It is clear that $N_u = N_u^c + N_u^d$, therefore, taking expectation on both sides, we have

$$\mathbb{E}\, N_u = \mathbb{E}\, N_u^c + \mathbb{E}\, N_u^d. \tag{3.1}$$

The two terms of the r.h.s. of Equation (3.1) are treated separately and by different methods. For the continuous crossings we recall that classical Rice Formula is based on Kac Counting Formula for the number of crossings of a C^1 function, and observe that when the function has jumps (and the value u is not one of the lateral limits) Kac Formula counts the number of continuous crossings through the level, ignoring the discontinuous ones. On the other hand, for the discontinuous crossings we use techniques from point processes theory.

We present now the main theorem of the first part of this monograph, which includes the case of non-Gaussian continuous processes $\mathcal{Z}$.

Theorem 3.1. *Let $\mathcal{Z}$ and $\mathcal{J}$ be two independent processes on $[0,T]$ and assume the following conditions*

1. *$\mathcal{Z}$ verifies the conditions A1, A2, A3 and A4 described above,*
2. *$\mathcal{J}$ is a pure jump process, as described above, with $\lambda < \infty$*

Let also $\mathcal{X}$ be defined by $X(t) = Z(t)+J(t)$. Then, the mean number of continuous and discontinuous crossings through the level u by the process $\mathcal{X}$ on the interval $[0,T]$ are given, respectively, by:

$$\begin{aligned}\mathbb{E}\, N_u^c &= \int_{[0,T]} \mathbb{E}\left(|X'(t)| \mid X(t) = u\right) p_{X(t)}(u)dt \\ &= \int_{[0,T]} \mathbb{E}\left(|Z'(t)| \mid X(t) = u\right) p_{X(t)}(u)dt,\end{aligned}$$

and

$$\mathbb{E}\, N_u^d = \mathbb{E} \iint_{[0,T]\times\mathbb{R}} \mathbb{I}\{(X(t^-) - u)(X(t^-) + y - u) < 0\} L(dt, dy),$$

where $\mathbb{I}A$ is the indicator (or characteristic) function of the set A and L is the compensating measure of the random counting measure generated on $[0,T]\times\mathbb{R}$ by the jump process $\mathcal{J}$. Similar formulas hold for the number of up and down crossings with self evident changes.

Some remarks are in order. First, note that when the jump process $\mathcal{J}$ vanishes, that is, if $J(t)=0$ almost surely for all $t\in[0,T]$, Theorem 3.1 reduces to Classical Rice Formula for the process $\mathcal{Z}$.

Next, observe that the random variable $J(t)$ does not need to have a density for each t, but, in case it does we have a more explicit result.

Corollary 3.1. *If $J(t)$ has a continuous density $p_{J(t)}(x)$ for $t\in[0,T]$ and $x\in\mathbb{R}$, we can also write*

$$\mathbb{E}\,N_u^c=\int_{[0,T]}dt\int_{\mathbb{R}}\mathbb{E}\,\left(|Z'(t)|\mid Z(t)=v\right)p_{Z(t)}(v)p_{J(t)}(u-v)dv.$$

Proof. Using the first equality in Lemma 3.1 in order to add the condition on the value of $Z(t)$, the integrand of the second expression in Theorem 3.1 for the number of continuous crossings becomes

$$\begin{aligned}\int_{\mathbb{R}}\mathbb{E}\,\left[|Z'(t)|\mid X(t)=u,Z(t)=v\right]p_{Z(t)|X(t)=u}(v)p_{X(t)}(u)dv\\ =\int_{\mathbb{R}}\mathbb{E}\,\left[|Z'(t)|\mid Z(t)=v,J(t)=u-v\right]p_{Z(t)|X(t)=u}(v)p_{X(t)}(u)dv\end{aligned}$$

where we used the fact that the events $\{X(t)=u,Z(t)=v\}$ and $\{Z(t)=v,J(t)=u-v\}$ are equivalent. Further, since $\mathcal{Z}$, and thus $\mathcal{Z}'$, is independent from $\mathcal{J}$, we can remove the condition $J(t)=u-v$ from the conditional expectation. Finally, by standard arguments, it follows that $p_{Z(t)|X(t)=u}(v)p_{X(t)}(u)=p_{Z(t),J(t)}(v,u-v)$. This proves the corollary. □

Finally, a careful analysis of the proof of Theorem 3.1 in Section 4 shows that the result in Theorem 3.1 holds true whenever the law of the process $\mathcal{Z}$, restricted to the subintervals $[\tau_i,\tau_{i+1}]$, conditioned to the paths of the jump process verifies the hypothesis $A1$ - $A4$. Besides, $A3$ can be weakened assuming that the product of the conditional expectation and the density of $X(t)$ (or $Z(t)$ in Corollary 3.1) is continuous.

We end this section specializing these results to the case where $\mathcal{Z}$ is a Gaussian process, here, the hypothesis of continuity of the densities and of the conditional expectation may be released since they follow from the conditions of non-degeneracy of the distribution of $Z(t)$, $t\in[0,T]$, and on the regularity of the paths. Besides, the ingredients in the formulas are computable explicitly, see the Appendix A in page 131.

Corollary 3.2. *Let $\mathcal{Z}$ be a Gaussian process with C^1 paths such that for every t the distribution of $Z(t)$ is non-degenerated, assume further that $\mathcal{J}$ is a pure jump process independent from $\mathcal{Z}$ with finite intensity of jumps. Then the result of Theorem 3.1 holds true.*

Corollary 3.3. *Under the hypothesis of Corollary 3.2, if $\mathcal{Z}$ has constant variance, then, the formula for the continuous crossings reduces to:*

$$\mathbb{E}\, N_u^c = \mathbb{E}\,|Z'(0)| \int_0^T p_{X(t)}(u)dt.$$

In particular, if $\mathcal{X}$ is a stationary process, last formula further reduces to

$$\mathbb{E}\, N_u^c = T\,\mathbb{E}\,|Z'(0)|p_{X(0)}(u).$$

Proof. This follows from the well known fact that for a centered Gaussian process, having constant variance implies the independence of the process and its derivative at each point. □

Remark 3.2. *Under the hypothesis of Corollary 3.3, let us define $p : \mathbb{R} \to \mathbb{R}$ by*

$$p(u) := \frac{1}{T} \int_0^T p_{X(t)}(u)\, dt.$$

It is easy to check that p is a probability density function. Thus, we can rewrite the formula in Corollary 3.3 as

$$\mathbb{E}\, N_u^c = T\,\mathbb{E}\,|Z'(0)|\, p(u).$$

Thus, we recover the nice fact that the mean number of continuous crossings through the level u by $\mathcal{X}$ is proportional to a probability density function evaluated at u. This expression is similar to Rice's original result, but it involves a (in principle) different density function.

In particular, if $\mathcal{X}$ is a stationary process, the density p reduces to that of $X(0)$, just as in the original formula due to Rice. Nevertheless, the process is not the same as in Rice's original result.

Furthermore, observe that in general, the density p is a mixture of the densities of the uni-dimensional laws of $X(t)$ for $t \in [0, T]$. Thus in the stationary case appears the density of X at any given time, and in the general case appears a mixture of these densities.

3.3 Generalization of Borovkov-Last formula

In this section we move from the main line of the chapter and consider a different class of stochastic processes, namely, the Piecewise Deterministic Markov Processes.

Borovkov & Last in [20] are interested in the continuous crossings through a fixed level u by a stationary Piecewise Deterministic Markov Process. Roughly speaking, a process $\mathcal{X}$ of this class, starts its evolution at a random position, then jumps a random quantity at random times but moves deterministically between jumps.

These processes are formally described by a general point process $(\tau_n, \xi_n)_n$, as described in Appendix B, see page 135, and a (non-random) rate function $\mu : \mathbb{R} \to \mathbb{R}$. More precisely, the process $\mathcal{X}$ has jumps at the epochs (τ_n), the magnitude of the n-th jump is ξ_n and on the interval $[\tau_n, \tau_{n+1})$, $X(t)$ follows the integral curve of μ with initial condition $X(\tau_n) = X(\tau_n^-) + \xi_n$.

Note that the jump part of the process is not independent from the continuous one.

Let $D_\mu = \{u : \mu(u) = 0\}$ be the set of critical levels. Observe that if $\mu(u) > 0$ (resp. $<$), the continuous crossings through the level u can only be up-crossings (resp. down-crossings).

Next theorem extends Borovkov-Last Formula to the non-stationary case.

Theorem 3.2. *Let $u \notin D_\mu$ and assume that μ and $p_{X(t)}$ are continuous w.r.t. x in a neighborhood of u and $t \in [0, T]$. Then*

$$\mathbb{E}\, N_u^c = |\mu(u)| \int_0^T p_{X(t)}(u) dt$$

Proof. For the levels $u \notin D_\mu$ we can apply Kac counting formula path wise for almost all paths of $\mathcal{X}$. In effect, the continuity of μ implies that the paths are of class C^1 between the jumps and that $X'(t) = \mu(X(t))$ for almost all $t \in [0, T]$. Since $X(t)$ has a density, the value u is not taken at the extremes of the interval neither at the jump points almost surely. Furthermore, by the continuity of this density, there are not tangencies at level u.

Now we take expectation on both sides of Kac Counting Formula and observe that the number of continuous crossings of the level $u \notin D_\mu$ is bounded by the number of jumps $+1$ of $\mathcal{X}$ in $[0, T]$. In effect, the sign of $\mu(u)$ determines the direction of the continuous crossings of u, so, between two continuous crossings must be a discontinuous one in the opposite direction, thus there is at most one

continuous crossing at each one of the intervals of the partition $\tau_0, \ldots, \tau_{\nu_T}, T$. Then, since ν_T in integrable, we may pass to the limit under the expectation sign:

$$\mathbb{E}\, N_u^c = \lim_{\delta \downarrow 0} \frac{1}{2\delta} \int_0^T \mathbb{E}\, \left[|X'(t)| \mathbb{I}_{\{|X(t)-u|<\delta\}} \right] dt.$$

Now, for each $t \in [0,T]$, with probability one t is not a jump epoch, then $X'(t)$ is a deterministic function of $X(t)$, namely $X'(t) = \mu(X(t))$, so the integrand is simply the expectation of a function of $X(t)$, therefore

$$\mathbb{E}\, N_u^c = \lim_{\delta \downarrow 0} \frac{1}{2\delta} \int_0^T \int_{u-\delta}^{u+\delta} |\mu(x)| p_{X(t)}(x) dx dt.$$

By the continuity of the integrand and the compactness of the domain we can pass the limit inside the integral w.r.t. t. Then, the result follows by the mean value theorem. □

As a corollary, when $\mathcal{X}$ is a stationary process, we obtain Borovkov-Last's Formula [20, Th. 3.1].

Corollary 3.4. *If in addition to the conditions of Theorem 3.2, the process $\mathcal{X}$ is stationary, then*

$$\mathbb{E}\, N_u^c = |\mu(u)| p(u),$$

where $p = p_{X(0)}$.

Remark 3.3. *Note that on this case the (net) number of discontinuous crossings, namely the difference of down and up crossings $D_u^d - U_u^d$, is related to the number of continuous crossings, actually we have:*

$$\mathbb{E}\, D_u^d - \mathbb{E}\, U_u^d = \mathbb{E}\, N_u^c + \operatorname{sgn}(\mu(u))(\mathbb{P}(X(T) < u) - \mathbb{P}(X(0) < u))$$

3.4 Proofs of Theorem 3.1

In this section we present the proof of Theorem 3.1. Actually, since the general hypothesis of the main theorem are quite restrictive, we present also an alternative proof when the process $\mathcal{Z}$ is Gaussian by the arguments of polygonal approximation.

3.4.1 General proof

First, we prepare some auxiliary results.

Lemma 3.1. *Let T, X, Y, Z be random variables, then*

$$\mathbb{E}\left(T \mid X = x, Y = y\right) = \int_{-\infty}^{\infty} \mathbb{E}\left(T \mid X = x, Y = y, Z = z\right) p_{Z|X=x,Y=y}(z) dz.$$

and

$$\mathbb{E}\left(T \mid X = x, Z = z\right) p_{X|Z=Z}(x) = \int_{-\infty}^{\infty} \mathbb{E}\left(T \mid X = x, Y = y, Z = z\right) p_{X,Y|Z=z}(x, y) dy.$$

Proof. These equalities follow directly from the properties of conditional expectation, see Petrov & Mordecki [64, page 215]. For example, for the first one take $F_2 = (X, Y)$ and $F_3 = (X, Y, Z)$. □

Proof of Theorem 3.1. Let us start with the formula for the mean number of continuous crossings.

In order to separate the different structures of the process, we compute the expectation by conditioning on the paths of the pure-jump process $\mathcal{J}$ and make use of the proof of Rice Formula on the non-Gaussian case on Azaïs & Wschebor [12, Th. 3.4].

Then, we condition on the number of jumps, $\nu_T = n$, on the jump instants, $\tau_k = t_k$, thus

$$\mathbb{E}\, N_u^c = \mathbb{E}\left(\mathbb{E}\left[N_u^c\left(\mathcal{X}, [0,T]\right) \mid \nu_T = n; \boldsymbol{\tau} = \mathbf{t}\right]\right) = \sum_{n=0}^{\infty} p_{\nu_T}(n) \int_{[0,T]^n} p_{\boldsymbol{\tau}}(\mathbf{t})\, \mathbb{E}\left[N_u^c\left(\mathcal{X}, [0,T]\right) \mid \nu_T = n; \boldsymbol{\tau} = \mathbf{t}\right] d\mathbf{t},$$

where we set $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$ and $\mathbf{t} = (t_1, \ldots, t_n)$.

Now, we look at the integrand, since the number of crossings is additive w.r.t. the interval, we split the interval $[0, T]$ as the union of the intervals $\mathcal{I}_k := [t_{k-1}, t_k)$, then

$$\mathbb{E}\left[N_u^c\left(\mathcal{X}, [0,T]\right) \mid \nu_T = n; \boldsymbol{\tau} = \mathbf{t}\right] = \sum_{k=1}^{n} \mathbb{E}\left[N_u^c\left(\mathcal{X}, \mathcal{I}_k\right) \mid \nu_T = n; \boldsymbol{\tau} = \mathbf{t}\right]$$

Conditioning further on the values of the jump process at the jump epochs, each term can be written, see Lemma (3.1), as

$$\int_{-\infty}^{\infty} \mathbb{E}\left[N_u^c(\mathcal{X},\mathcal{I}_k) \mid \nu_T = n; \boldsymbol{\tau} = \mathbf{t}; J(t_{k-1}) = y\right] p_{J(t_{k-1})|\nu_T=n;\boldsymbol{\tau}=\mathbf{t}}(y)dy.$$

Now, conditionally on $\nu_T = n; \boldsymbol{\tau} = \mathbf{t}$ and $J(\tau_{k-1}) = y$, the process $\mathcal{X}$ can be written as $\mathcal{Z} + y$ on $\mathcal{I}_k$. Since $\mathcal{Z}$ verifies the conditions $A1$, $A2$, $A3$ and $A4$ on $\mathcal{I}_k$ so does the process $\mathcal{Z} + y$, therefore we may apply Rice Formula, see Azaïs & Wschebor [12], on each interval under these conditions to obtain

$$\begin{aligned}\mathbb{E}\left[N_u^c(\mathcal{X},\mathcal{I}_k) \mid \nu_T = n; \boldsymbol{\tau} = \mathbf{t}; J(t_{k-1}) = y\right] \\ = \int_{\tau_{k-1}}^{\tau_k} \mathbb{E}\left[|Z'(t)| \mid X(t) = u, \nu_T = n; \boldsymbol{\tau} = \mathbf{t}, J(\tau_{k-1}) = y\right] \\ \cdot p_{X(t)|\nu_T=n;\boldsymbol{\tau}=\mathbf{t},J(\tau_{k-1})=y}(u)dt\end{aligned}$$

Each one of these expressions should be replaced in the previous one.

Finally, we have to integrate (the conditions), for notational simplicity, let us write $g(n,\mathbf{t},y) = \mathbb{E}\left[|Z'(t)| \mid X(t) = u, \nu_T = n; \boldsymbol{\tau} = \mathbf{t}, J(\tau_{k-1}) = y\right]$, $g(n,\mathbf{t}) = \mathbb{E}\left[|Z'(t)| \mid X(t) = u, \nu_T = n; \boldsymbol{\tau} = \mathbf{t}\right]$ and $g(n) = \mathbb{E}\left[|Z'(t)| \mid X(t) = u, \nu_T = n\right]$. Let us perform the integrals one by one, starting w.r.t. y, (use Fubini) then

$$\begin{aligned}\int_{\mathbb{R}} g(n,\mathbf{t},y) p_{X(t)|\nu_T=n;\boldsymbol{\tau}=\mathbf{t},J(\tau_k)=y}(u) p_{J(t_k)|\nu_T=n;\boldsymbol{\tau}=\mathbf{t}}(y)dy \\ = \int_{\mathbb{R}} g(n,\mathbf{t},y) p_{(X(t),J(\tau_k))|\nu_T=n;\boldsymbol{\tau}=\mathbf{t}}(u,y)dy \\ = g(n,\mathbf{t}) p_{X(t)|\nu_T=n;\boldsymbol{\tau}=\mathbf{t}}(u),\end{aligned}$$

Where we used Lemma (3.1). Now, we sum the integrals over k and integrate w.r.t. $\mathbf{t}$:

$$\begin{aligned}\int_0^T dt \int_{[0,T]^n} g(n,\mathbf{t}) p_{X(t)|\nu_T=n;\boldsymbol{\tau}=\mathbf{t}}(u) p_{\boldsymbol{\tau}|\nu_T=n}(\mathbf{t})d\mathbf{t} \\ = \int_0^T dt \int_{[0,T]^n} g(n,\mathbf{t}) p_{(X(t),\boldsymbol{\tau})|\nu_T=n}(u.\mathbf{t})d\mathbf{t} \\ = \int_0^T g(n) p_{X(t)|\nu_T=n}(u)dt.\end{aligned}$$

By the same arguments one can remove the condition on ν_T from the density in

the last integral. In effect

$$\sum_{n=0}^{\infty} p_{\nu_T}(n) \int_0^T g(n) p_{X(t)|\nu_T=n}(u)dt$$
$$= \int_0^T \sum_{n=0}^{\infty} \mathbb{E}\left[|Z'(t)| \mid X(t)=u, \nu_T=n\right] p_{X(t),\nu_T}(u,n)dt$$
$$= \int_0^T \mathbb{E}\left[|Z'(t)| \mid X(t)=u\right] p_{X(t)}(u)dt$$

where we used that $p_{X(t)|\nu_T=n}(u)p_{\nu_T}(n) = p_{X(t),\nu_T}(u,n)$. The result follows.

Now we proceed to the formula for the mean number of discontinuous crossings through level u.

Clearly, the process $\mathcal{X}$ only can have a discontinuous crossing through the level u at the jump epochs $\tau_n; n = 1, \ldots, \nu_T$, and the magnitude of the jump of $\mathcal{X}$ at each one of these points is ξ_n. Hence, we consider the marked point process $((\tau_k, \xi_k) : k \geq 0)$ associated to $\mathcal{J}$ on $[0,\infty) \times \mathbb{R}$, which defines a random counting measure $\mu(dt, dy)$, in terms of which we can write:

$$N_u^d = \sum_{k=1}^{\nu_T} \mathbb{I}\{(X(\tau_k^-) - u)(X(\tau_k^-) + \Delta X(\tau_k) - u) < 0\}$$
$$= \sum_{0 \leq t \leq T} \mathbb{I}\{(X(t^-) - u)(X(t^-) + \xi_{\nu_t} - u) < 0\}$$
$$= \int_{[0,T]\times\mathbb{R}} \mathbb{I}\{(X(t^-) - u)(X(t^-) + y - u) < 0\}\mu(dt, dy)$$

Is easy to see that this RCM has a compensating measure denoted by $L(dt, dy)$, see Jacobsen [42] or Appendix B in page 135. Taking expectations, conditional w.r.t. the continuous process $\mathcal{Z}$, on both sides we have

$$\mathbb{E}\, N_u^d = \mathbb{E}\left(\mathbb{E}\left[N_u^d \mid \mathcal{Z}\right]\right)$$
$$= \mathbb{E}\left(\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} \mathbb{I}\{(X(t^-) - u)(X(t^-) + y - u) < 0\}\mu(dt, dy) \mid \mathcal{Z}\right]\right)$$
$$= \mathbb{E} \int_{[0,T]\times\mathbb{R}} \mathbb{I}\{(X(t^-) - u)(X(t^-) + y - u) < 0\}L(dt, dy)$$

where we used that, conditioned on $\mathcal{Z}$, the integral is done w.r.t. the RCM μ associated with $\mathcal{J}$ which coincides with the integral w.r.t. the compensating measure $L(dt, dy)$ since the integrand is predictable, in fact it is a function of t^-. Finally we integrate with respect to $\mathcal{Z}$ and the result follows. □

3.4.2 Alternative proof for Gaussian $\mathbb{Z}$

As we have already mentioned, when $\mathcal{Z}$ is a Gaussian process things are easier, see Appendix A in page 131.

For instance, we can adapt the proof of Rice Formula via the arguments of polygonal approximation of the process in order to include the jumps. The only care we need to take is to approximate in this form only the continuous process $\mathcal{Z}$, and not the total process $\mathcal{X}$. Otherwise, each continuous or discontinuous crossing of the original process $\mathcal{X}$ would generate a (continuous) crossings of the polygonal approximating process.

The arguments of polygonal approximation have been used extensively by Leadbetter & Spaniolo [55] and also by Azaïs & Wschebor [12].

We prepare some definitions and auxiliary results first.

For $m \in \mathbb{N}$, let $\mathcal{Z}_m$ be the dyadic polygonal approximation of $\mathcal{Z}$ on $[0,T]$, that is, at the dyadic points $\frac{k}{m}T : k = 0, 1, \ldots, m$ the process $\mathcal{Z}_m$ coincides with $\mathcal{Z}$

$$Z_m\left(\frac{k}{m}T\right) = Z\left(\frac{k}{m}T\right),$$

and on the intervals $\left[\frac{k}{m}T, \frac{k+1}{m}T\right]$ the paths of $\mathcal{Z}_m$ are linear. It is easy to see that the process $\mathcal{Z}_m$ is also Gaussian.

Besides, define the approximating process $\mathcal{X}_m$ by

$$X_m(t) = Z_m(t) + J(t)$$

for $t \in [0,T]$.

Lemma 3.2. *We need the following facts.*

1. *$X_m(t)$ has a density for each t,*
2. *the map*
$$x \mapsto \mathbb{E}\left(Z'^{+}(t) \mid Z(t) = x\right) p_{Z(t)}(x)$$
is bounded on $\mathbb{R}$, actually it is bounded by $(c_1 + c_2|x|)p_{Z(t)}(x)$ with $c_1, c_2 > 0$.

Proof. 1. This follows from the convolution formula since $Z_m(t)$ and $J(t)$ are independent and have densities.

2. This follows from the Gaussian regression formula. In effect, for each $t \in [0,T]$ we have

$$\mathbb{E}\left[Z_m'^{+}(t) \mid Z_m(t) = x\right] = \mathbb{E}\left(\zeta(t) + c(t)x\right)$$

where $\zeta(t)$ is a centered Gaussian random variable with variance

$$var(\zeta(t)) = var(Z_m'^+(t)) - \frac{cov^2(Z_m'^+(t), Z_m(t))}{var(Z_m(t))}$$

and $c(t) = cov(Z_m'^+(t), Z_m(t))/var(Z_m(t))$. From the assumptions on non-degeneracy of the 1-dimensional distributions and of smoothness of the paths of the Gaussian process $\mathcal{Z}$ it follows that $var(\zeta(t))$ and $c(t)$ are bounded functions on $[0,T]$ and thus the claim is proved.

□

We turn now to the second proof of Theorem 3.1 in the case of Gaussian continuous part $\mathcal{Z}$. This is Corollary 3.2.

Proof of Corollary 3.2. The proof has two parts.

Let us begin with the formula for the mean number of discontinuous crossings. Observe that the same argument as in the proof of Theorem 3.1 carries on, note that the process $\mathcal{Z}$ plays a secondary role in this part of the proof.

We turn to the mean number of continuous crossings through the level u by the process $\mathcal{X}$. We follow the proof of Rice Formula on the Gaussian case in Azaïs & Wschebor [12, Th. 3.1].

Since the paths of the process $\mathcal{X}_m$ are almost surely piecewise C^1 and $X_m(t)$ has a density for all t, we can apply Kac counting Formula to count the number of continuous up-crossings of $\mathcal{X}_m$, we have

$$U_u^c(\mathcal{X}_m) = \lim_{\delta \downarrow 0} \frac{1}{2\delta} \int_{[0,T]} X_m'^+(t) \mathbb{I}_{\{|X_m(t)-u|<\delta\}} dt.$$

Furthermore, since the paths of $\mathcal{J}$ are piecewise constant it follows that, for almost every $t \in [0,T]$, $X_m'^+(t) = Z_m'^+(t)$, then

$$U_u^c(\mathcal{X}_m) = \lim_{\delta \downarrow 0} \frac{1}{2\delta} \int_{[0,T]} Z_m'^+(t) \mathbb{I}_{\{|X_m(t)-u|<\delta\}} dt.$$

For each realization of $\mathcal{X}_m$, the r.h.s. counts the number of up-crossings of the level u. Since the paths of this process are linear between the points of the partition formed by dyadic points and jump epochs, there can not be more than one up-crossing on each interval of this partition. Therefore, the r.h.s. is not larger than the number of points in the partition, namely, $2^m + \nu_T$ which is an integrable function in $\Omega \times [0,T]$.

Hence, we take expectation on both sides and apply the Dominated Convergence Theorem to interchange limit and expectation signs. Thus

$$\mathbb{E}\, U_u^c(\mathcal{X}_m) = \lim_{\delta \downarrow 0} \frac{1}{2\delta} \int_{[0,T]} \mathbb{E}\, \left[Z_m'^{+}(t) \mathbb{I}_{\{|X_m(t)-u|<\delta\}} \right] dt.$$

Let us focus now on the expectation appearing in the integrand. Since $(Z_m(t), Z_m'(t))$ is a Gaussian random vector for each t and is independent from $J(t)$, the conditional expectation has a continuous version, defined via Gaussian regression, Therefore we can write

$$\begin{aligned}\mathbb{E}\, U_u^c(\mathcal{X}_m) &= \lim_{\delta \downarrow 0} \frac{1}{2\delta} \int_{[0,T]} \iint_{\mathbb{R}^2} \mathbb{E}\, \left[Z_m'^{+}(t) \mathbb{I}_{\{|X_m(t)-u|<\delta\}} \mid Z_m(t) = x, J(t) = y \right] \\ &\qquad \cdot p_{Z_m(t)}(x) p_{J(t)}(y) dx dy dt \\ &= \lim_{\delta \downarrow 0} \frac{1}{2\delta} \int_{[0,T]} dt \int_{\mathbb{R}} p_{J(t)}(y) dy \int_{\{|x+y-u|<\delta\}} \mathbb{E}\, \left[Z_m'^{+}(t) \mid Z_m(t) = x \right] p_{Z_m(t)}(x) dx \\ &= \lim_{\delta \downarrow 0} \int_{[0,T]\times\mathbb{R}} f_\delta(t,y) \cdot p_{J(t)}(dy) dt\end{aligned}$$

where we use the independence of $\mathcal{Z}$ and $\mathcal{J}$ to remove the condition on $J(t)$. In the last line we denote

$$f_\delta(t,y) = \frac{1}{2\delta} \int_{u-y-\delta}^{u-y+\delta} \mathbb{E}\, \left[Z_m'^{+}(t) \mid Z_m(t) = x \right] p_{Z_m(t)}(x) dx.$$

Therefore, by Lemma 3.2 and since $Z_m(t)$ is a centered Gaussian random variable for each $t \in [0,T]$, it follows that the integrand is bounded by a constant (the first factor is bounded by $c_1 + c_2|x|$ for some constants c_1 and c_2, but the density function tends to zero very rapidly as $|x| \to \infty$, hence, the integrand tends to zero as $|x| \to \infty$, as it is continuous we conclude that it is bounded by a constant). hence, so is f_δ.

Since, the measure $p_{J(t)}(dy)dt$ is finite (in fact its total measure is T), we can apply Lebesgue's Dominated Convergence Theorem and change the limit and integral signs.

$$\lim_{\delta \downarrow 0} f_\delta(t,y) = \mathbb{E}\, \left[Z_m'^{+}(t) \mid Z_m(t) = u - y \right] p_{Z_m(t)}(u-y)$$

Then, doing a change of variables

$$\mathbb{E}\, U_u^c(\mathcal{X}_m) = \int_{[0,T]} dt \int_{\mathbb{R}} \mathbb{E}\, \left[Z_m'^{+}(t) \mid Z_m(t) = v \right] p_{Z_m(t)}(v) p_{J(t)}(u-v) dy$$

We have obtained so far the result for the polygonal approximation $\mathcal{X}_m$. Let us take the limit $m \to \infty$.

In the first place, it is easy to see that $U_u^c(\mathcal{X}_m)$ increases to $U_u^c(\mathcal{X})$, then the Monotone Convergence Theorem implies that $\mathbb{E}\, U_u^c(\mathcal{X}_m) \to_m \mathbb{E}\, U_u^c(\mathcal{X})$.

On the other hand, the integrand is a continuous function of $\mathbb{E}\, Z_m(t)$ and $var(Z_m(t))$, which converge uniformly to the mean and variance of $Z(t)$ respectively, so we can pass to the limit inside the integral and the limit is the same integral of the mean and variance of $Z(t)$.

This proves the Corollary. □

Chapter 4

Examples and application to the distribution of the maximum

In this chapter we present two examples of actual computation of the formulas introduced in the preceding chapter. Then we apply them to the study of the distribution of the maximum of the process on a compact time interval as the level grows to infinity. Furthermore, we compare which kind of crossings predominate as the level u tends to infinity. Finally, we study the distribution of the maximum as the level grows to infinity for a pure jump process.

4.1 Examples

In this section we present two examples. Remember that the process $\mathcal{X}$ is written as the sum of two independent processes $\mathcal{Z}$ and $\mathcal{J}$, representing the continuous part and the jumps respectively.

4.1.1 Stationary processes with 1-dimensional Gaussian distribution

In this example we assume that $\mathcal{Z} = (Z(t) : t \in [0, T])$ is a stationary, centered Gaussian process with C^1 paths. Such a process is completely described by its covariance function, see Azaïs & Wschebor [12]. The covariance function of $\mathcal{Z}$ is defined by $r(\tau) = \mathbb{E}\, Z(0)Z(\tau)$, without lose of generality, assume that $r(0) = 1/2$.

Let $\mathcal{J} = (J(t) : t \in [0, T])$ be a pure jump process constructed in the following way. Given a sequence of independent and identically distributed random

variables $(\xi_n : n \in \mathbb{N})$ with common centered Gaussian distribution with variance $1/2$, and $\rho \in \mathbb{R}$ such that $|\rho| < 1$, set $A_0 = \xi_0$ and, for $n \geq 1$, define

$$A_n = \rho A_{n-1} + \sqrt{1-\rho^2}\,\xi_n.$$

It is easy to check that $(A_n : n \in \mathbb{N})$ is a centered, stationary Gaussian sequence with variance $1/2$ and covariance between A_0 and A_n given by $\rho^n/2$.

Further, consider a (simple) Poisson process $\nu = (\nu_t : t > 0)$ with intensity $0 < \lambda < \infty$, independent from $(\xi_n : n \geq 0)$ and from $\mathcal{Z}$, that is, consider $(\tau_n : n \geq 0)$ such that $\tau_0 = 0$ and, for $n \geq 1$, $\tau_n - \tau_{n-1}$ are i.i.d. exponential random variables with intensity λ and let

$$\nu_t = \max\{n : \tau_n \leq t\}. \tag{4.1}$$

Define

$$J(t) = A_{\nu_t}.$$

Remark 4.1. *To define the process $\mathcal{J}$ in terms of kernels, let $P^{(n)}_{x_{n-1},\tau_{n-1}}$ be the exponential distribution with intensity λ (regardless of x_{n-1}, τ_{n-1}). Besides, let $\pi^{(n)}_{x_{n-1},\tau_n}$, representing the increments at the jump instant τ_n, be the normal distribution centered at $(\rho-1)x_{n-1}$ with variance $(1-\rho^2)/2$.*

We call such a process a Poisson-auto-regressive process (of order 1, covariance ρ and intensity λ).

Proposition 4.1. *A Poisson-auto-regressive process $\mathcal{J}$ is wide-sense stationary. Besides, the random variable $J(t)$ has a centered Gaussian distribution with variance $1/2$ for every t. The covariance between $J(t)$ and $J(t+\tau)$ is given by $e^{\lambda(1-\rho)\tau}/2$.*

Proof. We compute the distribution and the covariances of $J(t)$ conditioning on the number of jumps of $\mathcal{J}$:

$$\mathbb{P}(J(t) \leq x) = \sum_{n=0}^{\infty} p_{\nu_t}(n)\mathbb{P}(A_n \leq x) = \mathbb{P}(A_0 \leq x),$$

since A_n has centered Gaussian distribution with variance $1/2$ for each n, hence, so does $J(t)$ for all t. In particular, $J(t)$ is centered for all t.

For the second moments we have:

$$\begin{aligned}
&\mathbb{E}\, J(t)J(t+\tau) \\
&\quad = \sum_{n,k=0}^{\infty} p_{\nu_t}(n)\mathbb{P}(\nu(t,t+\tau]=k)\mathbb{E}\left(J(t)J(t+\tau) \mid \nu_t = n, \nu(t,t+\tau]=k\right) \\
&\quad = \sum_{n,k=0}^{\infty} p_{\nu_t}(n)p_{\nu_\tau}(k)\mathbb{E}\left(A_n A_{n+k}\right) = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\sum_{k=0}^{\infty} e^{-\lambda\tau}\frac{(\lambda\tau)^k}{k!}\frac{\rho^k}{2} \\
&\quad = \frac{1}{2}e^{-\lambda\tau}e^{\lambda\rho\tau},
\end{aligned}$$

where $\nu(t,t+\tau]$ denotes the number of jump epochs on the interval $(t,t+\tau]$. □

Next theorem gives the mean number of continuous and discontinuous crossings through the level u by the process $\mathcal{X}$ on the interval $[0,T]$. For the sake of notational simplicity we consider only the up-crossings, but the case of down-crossings is completely analogous.

Theorem 4.1. *Let $\mathcal{X} = \mathcal{Z} + \mathcal{J}$ with $\mathcal{Z}, \mathcal{J}$ independent processes such that $\mathcal{Z}$ is a stationary, centered Gaussian process with $r(0) = 1/2$ and C^1 paths, and $\mathcal{J}$ a Poisson-auto-regressive process. Then*

$$\mathbb{E}\, U_u = T\sqrt{\frac{\lambda_2}{2\pi}}\,\varphi(u) + \lambda T\, p_\rho(u),$$

where φ stands for the standard Gaussian density function, λ_2 is the second spectral moment of $\mathcal{Z}$, and $p_\rho(u)$ is the probability that a two dimensional, centered, Gaussian vector with unit variances and covariance $(1+\rho)/2$ belongs to the set $(-\infty,u)\times(u,\infty)$.

Proof. We begin considering the mean number of continuous crossings, as $\mathcal{Z}$ is a Gaussian and stationary process, we can apply Corollary 3.3. Besides, by Proposition 4.1, the density $p_{X(t)} = p_{X(0)} = \varphi$ for all t, therefore

$$\mathbb{E}\, U_u^c = T\,\mathbb{E}\, Z'^{+}(0)\,\varphi(u).$$

Finally, an elementary computation shows that $\mathbb{E}\, Z'^{+}(0) = \sqrt{\lambda_2/2\pi}$. This gives the first term of $\mathbb{E}\, U_u$.

Let us consider now the discontinuous up-crossings. The compensating measure of the point process $(\tau_n, \Delta A_n)_n$, $\Delta A_n = A_n - A_{n-1}$, is $\lambda dt\, F(dy)$, where F is the Gaussian distribution centered at $(\rho-1)A_{\nu_{t^-}}$ and with variance $(1-\rho^2)/2$,

see Jacobsen [42, eq. 4.64]. Hence

$$\mathbb{E}\, U_u^d = \mathbb{E} \int_0^T \int_{\mathbb{R}} \mathbb{I}\{X(t^-) < u, X(t^-) + y > u\} \lambda dt\, F(dy)$$
$$= \lambda \mathbb{E} \int_0^T dt \int_{\mathbb{R}} \mathbb{I}\{X(t^-) < u, X(t^-) + y > u\} F(dy).$$

Actually F is the distribution of ΔA_{ν_t} conditioned on the random vector $(Z(t), A_{\nu_{t^-}})$, so

$$\mathbb{E}\, U_u^d = \lambda \mathbb{E} \int_0^T \mathbb{P}\left(X(t^-) < u, X(t^-) + \Delta A_{\nu_t} > u \mid Z(t), A_{\nu_{t^-}}\right) dt$$
$$= \lambda \mathbb{E} \int_0^T \sum_{n=0}^{\infty} p_{\nu_t}(n) \mathbb{P}\left(Z(t) + A_n < u, Z(t) + A_n + \Delta A_{n+1} > u \mid Z(t), A_n\right) dt$$
$$= \lambda \int_0^T \sum_{n=0}^{\infty} p_{\nu_t}(n) \mathbb{E}\, \mathbb{P}\left(Z(t) + A_n < u, Z(t) + A_n + \Delta A_{n+1} > u \mid Z(t), A_n\right) dt$$
$$= \lambda \int_0^T \sum_{n=0}^{\infty} p_{\nu_t}(n) \mathbb{P}\left(Z(t) + A_n < u, Z(t) + A_n + \Delta A_{n+1} > u\right) dt$$

where we have conditioned on the number of jumps and used Fubini's theorem. By the stationarity of $\mathcal{Z}$ and $\mathcal{J}$, the last probability does not depend on t and n, hence

$$\mathbb{E}\, U_u^d = \lambda \mathbb{P}\left(Z(0) + A_0 < u, Z(0) + A_0 + \Delta A_1 > u\right) \int_0^T \sum_{n=0}^{\infty} p_{\nu_t}(n) dt$$
$$= \lambda T \mathbb{P}\left(Z(0) + A_0 < u, Z(0) + A_0 + \Delta A_1 > u\right).$$

Now, observe that $A_0 + \Delta A_1 = A_1 = \rho\xi_0 + \sqrt{1-\rho^2}\xi_1$ and that the vector $(Z(0)+A_0, Z(0)+A_1)$ has centered Gaussian distribution with variances one and covariance $(1+\rho)/2$, therefore, the latter probability equals $p_\rho(u)$. This gives the second term and completes the proof. □

After computing the mean number of continuous and discontinuous up-crossings through the level u for this family of processes, we can ask which kind of (up-)crossings predominate when the level grows to infinity. Next result answers this question.

Corollary 4.1. *As $u \to \infty$, we have*

$$\lim \frac{\mathbb{E}\, U_u^d}{\mathbb{E}\, U_u^c} = 0.$$

In words, the continuous (up-)crossings predominate, on average, for high levels.

Proof. Let (X, Y) be a two dimensional centered Gaussian vector with $\mathbb{E}\, X^2 = \mathbb{E}\, Y^2 = 1/2$ and $\mathbb{E}\, XY = (1+\rho)/2$, then

$$p_\rho(u) = \mathbb{P}(X < u, Y > u) \leq \mathbb{P}(Y > u).$$

Then

$$p_\rho(u) \leq 1 - \Phi(u) \leq \frac{1}{u}\varphi(u) = o(\varphi(u)).$$

The result follows. □

Remark 4.2. *The processes $\mathcal{Z}$ and $\mathcal{J}$ contribute in $1/2$ to the variance of $\mathcal{X}$, more generally we can define*

$$\mathcal{X} = \alpha\mathcal{Z} + \sqrt{1-\alpha^2}\mathcal{J}$$

for $\alpha \in [0,1]$ and $\mathcal{Z}, \mathcal{J}$ as above but normalized in order to have variance one, namely, $r(0) = 1$ and $\xi_n : n \geq 0$ are i.i.d. standard Gaussian random variables. Hence,

$$\begin{aligned}\mathbb{E}\, N_u^c(\mathcal{X}) = T\,\mathbb{E}\,|X'(0)|\, p_{X(0)}(u) &= T\sqrt{\frac{2\lambda_2(\alpha Z(0))}{\pi}}\varphi(u) \\ &= \alpha T\sqrt{\frac{2\lambda_2(Z(0))}{\pi}}\varphi(u) = \alpha T\,\mathbb{E}\,|Z'(0)|\,\varphi(u) \\ &= \alpha\,\mathbb{E}\, N_u^c(\mathcal{Z}).\end{aligned}$$

Therefore, the mean number of continuous crossings of $\mathcal{X}$ is proportional to that of the continuous process $\mathcal{Z}$, the constant of proportionality being α, that is, the standard deviation of the first summand $\alpha\mathcal{Z}$.

4.1.2 Stationary Gaussian continuous process plus CPP

In this example we consider a centered, stationary, Gaussian process $\mathcal{Z}$ with $r(0) = 1$ and C^1 paths and an independent Compound Poisson Process (CPP), $\mathcal{J}$ with finite intensity λ and standard Gaussian jumps.

The process $\mathcal{J}$ is defined by the kernels $P^{(n)}_{x_{n-1},\tau_{n-1}}$, the exponential distribution of intensity $0 < \lambda < \infty$ and $\pi^{(n)}_{x_{n-1},\tau_n}$, representing the increments, the standard Gaussian distribution. We can also define $\mathcal{J}$ in terms of random variables as follows, let $(\nu_t : t \geq 0)$ be a simple Poisson Process with intensity $\lambda > 0$, as

defined in Equation (4.1), and let $(\xi_n)_{n\in\mathbb{N}}$ be independent standard Gaussian random variables independent from (ν_t), then for $t \geq 0$ define

$$J(t) := \sum_{n=1}^{\nu_t} \xi_n.$$

Thus, $J(t)$ is the sum of a random number of standard Gaussian random variables.

It is well known, see Jacobsen [42] for instance, that the compensating measure of the CPP is $\lambda dt\, \Phi(dx)$, where Φ is the standard Gaussian distribution. In particular, the compensating measure is deterministic.

Denote by φ_n the density function of a centered Gaussian random variable with variance n, in particular, $\varphi_1 = \varphi$ is the standard Gaussian density. Recall that $\varphi_n * \varphi_m = \varphi_{n+m}$, where $*$ stands for the convolution.

Next theorem gives the mean number of up-crossings through u by $\mathcal{X}$. The mean number of down-crossings is analogue.

Theorem 4.2. *For the process defined above we have*

$$\mathbb{E}\, U_u^c = T\sqrt{\frac{\lambda_2}{2\pi}}\, p(u)$$
$$\mathbb{E}\, U_u^d = \lambda T \int_{-\infty}^{u} \overline{\Phi}(u-x)\, p(x)dx,$$

where $\overline{\Phi}(u) = 1-\Phi(u)$, λ_2 *is the second spectral moment of* $\mathcal{Z}$ *and* $p = \sum_{n=1}^{\infty} p_n\varphi_n$ *with* $p_n := \frac{1}{\lambda T}\mathbb{P}\{\nu_T \geq n\}$.

Remark 4.3. *In spite of the notation, the mean number of continuous up-crossings, does depend on* λ *through the density function* p*, which has expectation* 0 *and variance* $\lambda T/2$.

Remark 4.4. *Observe that the mean number of continuous crossings has the usual form, that is, is proportional to a probability density function evaluated at the level* u*; besides, the constant is the same as in the original Rice Formula. See Remark 3.2.*

Proof. We begin with the continuous crossings. Using the formula in Remark 3.1, we have

$$\mathbb{E}\, U_u^c = \int_0^T dt \int_{-\infty}^{\infty} \mathbb{E}\, \left(Z'^{+}(t) \mid Z(t) = v\right) p_{Z(t)}(v) p_{J(t)}(u-v)dv$$
$$= \mathbb{E}\, Z'^{+}(0) \int_0^T dt \int_{-\infty}^{\infty} p_{Z(0)}(v) p_{J(t)}(u-v)dv,$$

where we used the stationarity of $\mathcal{Z}$ as in Corollary 3.3. Recall that $\mathbb{E}\, Z'^{+}(0) = \sqrt{\lambda_2/2\pi}$. Now, by Lemma 4.1 we decompose $p_{J(t)}$, so

$$\begin{aligned}\mathbb{E}\, U_u^c &= \sqrt{\frac{\lambda_2}{2\pi}} \int_0^T dt \sum_{n=0}^{\infty} p_{\nu_t}(n) \int_{-\infty}^{\infty} p_{Z(0)}(v)\varphi_n(u-v)dv \\ &= \sqrt{\frac{\lambda_2}{2\pi}} \int_0^T dt \sum_{n=0}^{\infty} p_{\nu_t}(n)(p_{Z(0)} * \varphi_n)(u) \\ &= \sqrt{\frac{\lambda_2}{2\pi}} \sum_{n=0}^{\infty} \varphi_{n+1}(u) \int_0^T p_{\nu_t}(n)dt.\end{aligned}$$

This integral is computed in the second item of Lemma 4.1, we get

$$\begin{aligned}\mathbb{E}\, U_u^c &= T\sqrt{\frac{\lambda_2}{2\pi}} \sum_{n=0}^{\infty} \frac{1}{\lambda T} \mathbb{P}\left(\nu_T \geq n+1\right) \varphi_{n+1}(u) \\ &= T\sqrt{\frac{\lambda_2}{2\pi}} \sum_{n=0}^{\infty} p_{n+1}\varphi_{n+1}(u) = T\sqrt{\frac{\lambda_2}{2\pi}} \sum_{n=1}^{\infty} p_n\varphi_n(u) \\ &= T\sqrt{\frac{\lambda_2}{2\pi}}\, p(u)\end{aligned}$$

which gives the first part of the result.

Now, we turn to the discontinuous crossings

$$\begin{aligned}\mathbb{E}\, U_u^d &= \mathbb{E} \int_0^T \int_{-\infty}^{\infty} \mathbb{I}\{X(t^-) < u; X(t^-) + y > u\} L(dt, dy) \\ &= \mathbb{E} \int_0^T \int_{-\infty}^{\infty} \mathbb{I}\{X(t^-) < u; X(t^-) + y > u\} \lambda dt\, \Phi(dy),\end{aligned}$$

where, as we said before, Φ is the distribution of the jumps. Since the compensating measure is deterministic we have

$$\begin{aligned}\mathbb{E}\, U_u^d &= \lambda \int_0^T \int_{-\infty}^{\infty} \mathbb{E}\, \mathbb{I}\{X(t^-) < u; X(t^-) + y > u\} dt\, \Phi(dy) \\ &= \lambda \int_0^T \int_{-\infty}^{\infty} \mathbb{P}\big(X(t^-) < u; X(t^-) + y > u\big) dt\, \Phi(dy) \\ &= \lambda \int_0^T \mathbb{P}\big(X(t^-) < u; X(t^-) + \xi > u\big) dt,\end{aligned}$$

being ξ a standard Gaussian random variable independent from $X(t^-)$. Now, we condition on $X(t^-)$, since, for fixed, t almost surely $J(t^-) = J(t)$ we may use the

same computations as in the computation of $\mathbb{E}\, N_u^c$:

$$\begin{aligned}\mathbb{E}\, U_u^d &= \lambda \int_0^T \int_{-\infty}^u \mathbb{P}\big(\xi > u - x \mid X(t^-) = x\big)\, p_{X(t^-)}(x) dx dt \\ &= \lambda \int_{-\infty}^u \overline{\Phi}(u-x) dx \int_0^T p_{X(t^-)}(x) dt \\ &= \lambda T \int_{-\infty}^u \overline{\Phi}(u-x) p(x) dx.\end{aligned}$$

This completes the proof. □

Next, we compare the mean numbers of continuous and discontinuous up-crossings through the level u by $\mathcal{X}$ as $u \to \infty$.

Corollary 4.2. *As $u \to \infty$ the mean numbers of continuous and discontinuous up-crossings through the level u are of the same order. More precisely*

$$\lim_{u\to\infty} \frac{\lambda}{\sqrt{\lambda_2}}\, \mathbb{E}\, U_u^c \le \lim_{u\to\infty} \mathbb{E}\, U_u^d \le \lim_{u\to\infty} \lambda \sqrt{\frac{2\pi}{\lambda_2}}\, \mathbb{E}\, U_u^c$$

Proof. We bound from above and from below $\mathbb{E}\, U_u^d$. First, we have

$$\begin{aligned}\int_{-\infty}^u \overline{\Phi}(u-y) p(y) dy &= \int_0^\infty \overline{\Phi}(y) p(u-y) dy \\ &= \int_0^{2u} \overline{\Phi}(y) p(u-y) dy + \int_{2u}^\infty \overline{\Phi}(y) p(u-y) dy \\ &\ge \int_0^{2u} \overline{\Phi}(y) p(u-y) dy \ge p(u) \int_0^{2u} \overline{\Phi}(y) dy,\end{aligned}$$

where we used that p is a mixture of centered Gaussian densities and thus it is an even function that decreases on $[0, \infty)$. Furthermore

$$\lim_{u\to\infty} \int_0^{2u} \overline{\Phi}(y) dy = \int_0^\infty \overline{\Phi}(y) dy = \mathbb{E}\, \xi^+ = \sqrt{\frac{2}{\pi}}$$

where we used the well known formula for the expectation of a non-negative random variable in terms of its distribution.

Therefore

$$\mathbb{E}\, U_u^d \ge \lambda T\, p(u) \int_0^\infty \overline{\Phi}(y) dy \underset{u\to\infty}{\sim} \frac{\lambda T}{\sqrt{2\pi}} p(u).$$

On the other hand,

$$\mathbb{E}\, U_u^d \;=\; \lambda T \sum_{n=1}^\infty p_n \int_0^\infty \overline{\Phi}(u-y)\varphi_n(y) dy \;\le\; \lambda T \sum_{n=1}^\infty p_n \int_{-\infty}^\infty \overline{\Phi}(u-y)\varphi_n(y) dy.$$

Note that the last integral can be seen as the convolution, for the tail of the distribution function, of two (independent) random variables, say $Z \sim \Phi$ and $V_n \sim \varphi_n$. Then we can write the last integral as $\mathbb{P}(Z + V_n > u)$. Furthermore, since Φ is the Gaussian standard distribution and φ_n is the Gaussian density with zero mean and variance n, this probability equals $\mathbb{P}(V_{n+1} > u) = \overline{\Phi}(u/\sqrt{n+1})$. Thus

$$\begin{aligned}\mathbb{E}\, U_u^d &\leq \lambda T \sum_{n=1}^{\infty} p_n \overline{\Phi}\left(\frac{u}{\sqrt{n+1}}\right) \leq \lambda T \sum_{n=1}^{\infty} p_n (n+1)\varphi_{n+1}(u) \\ &= \lambda T (p(u) + \delta(u)),\end{aligned}$$

where $\delta(u) := \sum_{n=1}^{\infty} p_n(n+1)\varphi_{n+1}(u) - p(u)$. In order to obtain the desired result, it suffices to prove that $\delta(u) \to 0$ as $u \to \infty$. In effect

$$|\delta(u)| \leq \left| \sum_{n=1}^{\infty} \left[p_n(n+1) - p_{n+1} \right] \varphi_{n+1}(u) \right| + |p_1 \varphi_1(u)|$$

It is clear that the second term of the r.h.s tends to zero when $u \to \infty$.

Let us look at the first term, note that the sum $\sum_{n=1}^{\infty} p_n(n+1)$ is finite since it is, roughly speaking, the second factorial moment of the Poisson distribution. In effect

$$\begin{aligned}\sum_{n=1}^{\infty} p_n(n+1) &= 1 + \frac{1}{\lambda T} \sum_{n=1}^{\infty} n \mathbb{P}(\nu_T \geq n) = 1 + \frac{1}{\lambda T} \sum_{n=1}^{\infty} n \sum_{k=n}^{\infty} p_{\nu_T}(k) \\ &= 1 + \frac{1}{\lambda T} \sum_{k=1}^{\infty} p_{\nu_T}(k) \sum_{n=1}^{k} n = 1 + \frac{1}{2\lambda T} \sum_{k=1}^{\infty} k(k-1) p_{\nu_T}(k)\end{aligned}$$

where we used Lemma 4.1 in the first equality and standard computations in the remaining ones.

Thus, the first term is a mixture of Gaussian densities φ_n (times a constant), hence it tends to zero. Then $\delta(u) \to 0$ as $u \to \infty$ and

$$\mathbb{E}\, U_u^d \leq \lambda T (p(u) + \delta(u)) \underset{u \to \infty}{\sim} \lambda T p(u).$$

Now, putting together the two obtained bounds for $\mathbb{E}\, U_u^d$ and using Theorem 4.2 the desired result follows. □

Remark 4.5. *Note that in this case the discontinuous crossings through the level u are not negligible w.r.t. the continuous crossings when u tends to infinity, in contrast with the situation in the example of the previous section.*

We end with an auxiliary lemma.

Lemma 4.1. *Consider a CPP $\mathcal{J} = (J(t) : t \geq 0)$ such that $J(t) = \sum_{n=0}^{\nu_t} \xi_n$ with (ν_t) a simple Poisson Process with intensity λ and ξ_n standard Gaussian for all n, then*

1. *the density of the CPP can be written as:*

$$p_{J(t)}(x) = \sum_{n=1}^{\infty} p_{\nu_t}(n)\varphi_n(x).$$

2. *The integral of the density of ν_t w.r.t. t verifies*

$$\int_0^T p_{\nu_t}(n)dt = \frac{1}{\lambda}\mathbb{P}(\nu_T \geq n+1) = \frac{1}{\lambda}\mathbb{P}(\tau_{n+1} \leq T)$$

3. *If $p_n = \frac{1}{\lambda T}\mathbb{P}(\nu_T \geq n)$, then $\sum_{n=1}^{\infty} p_n = 1$*

As usual we denote the time epochs of $\mathcal{J}$ by τ_n, thus the number of jump epochs in $[0,t]$ is ν_t.

Proof. 1. Conditioning on the value of ν_t we have:

$$F_{J(t)}(x) = \mathbb{P}(J(t) \leq x) = \sum_{n=0}^{\infty} p_{\nu_t}(n)\mathbb{P}(J(t) \leq x \mid \nu_t = n).$$

Now, conditioned on the event $\nu_t = n$, $J(t)$ is the sum of n independent standard Gaussian random variables, hence, the conditional probability in the r.h.s. of the latter equation is the distribution of a centered normal random variable with variance n. The result follows taking derivatives on both sides.

2. By definition, $p_{\nu_t}(n)$, as a function of t, is equal to the density function of the Gamma distribution, with parameters λ, $n+1$, divided by λ, furthermore, it is well known, see Petrov & Mordecki [64], that this is the distribution of τ_{n+1}, hence

$$\int_0^T p_{\nu_t}(n)dt = \frac{\mathbb{P}(\tau_{n+1} \leq T)}{\lambda}.$$

The result follows since the events $\{\tau_{n+1} \leq T\}$ and $\{\nu_T \geq n+1\}$ coincide.

3. It follows directly from the facts that ν_T has Poisson distribution with mean λT and that for a non-negative integer valued random variable X: $\mathbb{E}\,X = \sum_{n=1}^{\infty} \mathbb{P}(X \geq n)$. □

4.2 Application to the distribution of the maximum on a compact time interval

As we mentioned in the Introduction and in the Preliminaries, the level crossing counting problem is intimately related to that of studying the distribution of the maximum of the process.

We now use Rice Formula to get upper and lower bounds, and an equivalence, for the tail of the distribution of the maximum of the process $\mathcal{X}$ with Gaussian stationary continuous part and Poisson-auto-regressive jump part introduced in Section 3.1. Recall that for all t the variance of $Z(t)$ and $J(t)$ is a half, yielding a total variance of one for $X(t)$.

Thus, we consider the maximum random variable of $\mathcal{X}$

$$M(T) = \max\{X(s) : 0 \leq s \leq T\}.$$

Our departing point is the elementary relation

$$\{M(T) > u\} = \{X(0) > u\} \uplus \{X(0) < u, U_u \geq 1\},$$

where $\uplus$ denotes the disjoint union. The upper bound is obtained using Markov's Inequality, it follows that

$$\mathbb{P}(M(T) > u) \leq \mathbb{P}(X(0) > u) + \mathbb{P}(U_u \geq 1) \leq \mathbb{P}(X(0) > u) + \mathbb{E}\, U_u. \tag{4.2}$$

On the other hand, since U_u is a non-negative integer valued random variable it satisfies the inequality $U_u \geq U_u - \frac{1}{2}U_u(U_u - 1)$, thus

$$\begin{aligned}\mathbb{P}(M(T) > u) &= \mathbb{P}(X(0) > u) + \mathbb{P}(U_u \geq 1) - \mathbb{P}(U_u \geq 1, X(0) > u) \\ &\geq \mathbb{P}(X(0) > u) + \mathbb{E}\, U_u - \frac{1}{2}\mathbb{E}\, U_{u.[2]} - \mathbb{P}(U_u \geq 1, X(0) > u),\end{aligned} \tag{4.3}$$

where $a_{[2]} = a(a-1)$ is the Pochammer symbol and $U_{u.[2]} = (U_u)_{[2]}$

The following theorem contains the upper bound for the tail of the distribution of the maximum of $\mathcal{X}$ on the interval $[0, T]$.

Theorem 4.3. *The probability that the maximum exceeds the level u verifies*

$$\mathbb{P}(M(T) > u) \leq \overline{\Phi}(u) + T\sqrt{\frac{\lambda_2}{2\pi}}\varphi(u) + \lambda T p_\rho(u), \tag{4.4}$$

where $\overline{\Phi}(u) = 1 - \Phi(u)$ and $p_\rho(u)$ is defined in Theorem 4.1.

Proof. This is a direct consequence of Theorem 4.1 and Formula (4.2). □

Remark 4.6. *Furthermore, if we denote the r.h.s. of Inequality (4.4) by $rhs(u)$, then, by Corollary 4.1 we have*

$$rhs(u) \underset{u\to\infty}{\sim} T\sqrt{\frac{\lambda_2}{2\pi}}\,\varphi(u).$$

The goal of the rest of this section is to show that this upper bound is sharp. In order to do that, we use the lower bound given in Inequality (4.3) for the tail of the distribution of the maximum $M(T)$. So, we have to deal with the second moment of the number up-crossings, which amounts to say that we have to deal with second order Rice Formula. In that sense, we give formulas for the second moment of the continuous up-crossings and for the second moment of discontinuous up-crossings.

Theorem 4.4. *Let the processes $\mathcal{Z}, \mathcal{X}$ and $\mathcal{J}$ be as in Corollary 3.3. If in addition $\mathcal{J}$ is a Poisson-auto-regressive process, and $\mathcal{Z}$ verifies that $r(\tau) \neq \pm 1/2$ for all τ and the Geman condition:*

$$\int \frac{\theta'(\tau)}{\tau^2} d\tau \text{ converges at } \tau = 0, \tag{4.5}$$

where θ is defined by

$$r(\tau) = 1 - \frac{\lambda_2}{2}\tau^2 + \theta(\tau) \tag{4.6}$$

Then

$$\mathbb{P}(M(T) > u) = \overline{\Phi}(u) + T\sqrt{\frac{\lambda_2}{2\pi}}\,\varphi(u) + o(\varphi(u))$$

where $f(u) = o(\varphi(u))$ means, as usual, that $f(u)/\varphi(u) \to 0$ as $u \to \infty$.

Remark 4.7. *Geman condition was originally introduced in 1967 as a sufficient condition for the finiteness of the second moment of the number of zero crossings for a stationary Gaussian process by Cramér & Leadbetter [23]. In 1972 Geman [36] proved that this condition is also a necessary condition. Recently, Kratz & León [50] extended this result for the levels other than zero.*

Proof. It suffices to show that the additional terms in Inequality (4.3) w.r.t. Inequality (4.2) are $o(\varphi(u))$. Let us begin with the second factorial moment of the number of up-crossings $\frac{1}{2}\mathbb{E}\,U_{u,[2]}$.

Since

$$U_{u,[2]} = U_u(U_u - 1) = U^c_{u,[2]} + U^d_{u,[2]} + 2\,U^c_u\,U^d_u,$$

the proof is divided in several parts, considering separately each one of the resulting terms.

Claim 4.1. *We have* $\mathbb{E}\, U^c_{u,[2]} = o(\varphi(u))$.

We prove the Claim in three steps.

Step 1. We begin with a Rice-type formula for the second moment of U^c_u.

$$\mathbb{E}\, U^c_{u.[2]} = \int_0^T \int_0^T \sum_{m,n=0}^{\infty} \mathbb{E}\left[Z'^+(s)Z'^+(t) \mid X(s) = X(t) = u, \nu_s = m, \nu_{t-s} = n\right] \cdot p_{X(s),X(t),\nu_s,\nu_{t-s}}(u,u,m,n)dsdt. \quad (4.7)$$

We adapt the proof of Rice Formula for the factorial moments in Azaïs & Wschebor [12, Th. 3.1]. As usual the diagonal plays an important role.

Let C_u be the set of continuous up-crossings of $\mathcal{X}$ through the level u on the interval $[0,T]$, $C_u^2 = C_u \times C_u$ and for any Borel set J in $[0,T]^2$ let

$$\mu(J) = \#(C_u^2 \cap J).$$

It is easy to check that $U^c_{u.[2]} = \mu([0,T]^2 \setminus \Delta)$, where Δ is the diagonal, that is, $\Delta = \{(s,t) \in [0,T]^2 : s = t\}$.

Take J_1 and J_2 disjoint intervals in $[0,T]$ and let $J = J_1 \times J_2$, i.e: $J \subset [0,T]^2 \setminus \Delta$, then

$$\begin{aligned}\mu(J) &= U_u(J_1) \cdot U_u(J_2)\\ &= \lim_{\delta\to 0} \frac{1}{(2\delta)^2} \int_{J_1} Z'^+(s)\mathbb{I}\{|X(s)-u| < \delta\}ds \cdot \int_{J_2} Z'^+(t)\mathbb{I}\{|X(t)-u| < \delta\}dt\\ &= \lim_{\delta\to 0} \frac{1}{(2\delta)^2} \iint_J Z'^+(s)Z'^+(t)\mathbb{I}\{|X(s)-u| < \delta, |X(t)-u| < \delta\}dsdt,\end{aligned}$$

where we applied Kac Formula on each interval, and noted that for δ small enough the quantity in the limit becomes constant, so we can use the same mute variable δ in both limits.

Now, the idea is to take expectation on both sides, but we have to proceed with some care if we want to get an equality instead of just an upper bound, the latter can readily be obtained using Fatou's Lemma. To get the former, we can approximate the process $\mathcal{Z}$ by the dyadic polygonal process as in the proof of Theorem 3.1, the same arguments as in that proof give the necessary dominations to pass the expectation inside the integral sign, and to show that the expressions for the dyadic polygonal approximation converge to those of the original process. In spite of this formal argument, we skip the details of the polygonal approximation and write the conditioning for the original process.

Then, let us condition on the number of jumps of the process in the intervals $[0,s]$, $[0,t]$ with $s \in J_1$, $t \in J_2$ and on the values of the process $\mathcal{X}$ at these points, that is (say that $s<t$)

$$\mathbb{E}\, Z'^{+}(s)Z'^{+}(t)\mathbb{I}\{|X(s)-u|<\delta, |X(t)-u|<\delta\} =$$
$$\int_{u-\delta}^{u+\delta}\int_{u-\delta}^{u+\delta}\sum_{m,n=0}^{\infty}\mathbb{E}\left[Z'^{+}(s)Z'^{+}(t) \mid X(s)=x, X(t)=y, \nu_s=m, \nu_t=m+n\right]$$
$$p_{X(s),X(t)|\nu_s=m,\nu_t-\nu_s=n}(x,y)p_{\nu_s}(m)p_{\nu_t}(m+n)dxdy$$

Observe that, under these conditions $X(s)=Z(s)+A_m$ and $X(t)=Z(t)+A_{m+n}$ are jointly Gaussian random variables, thus the conditional probability and the conditional expectation are well defined, the latter may be computed via Gaussian regression. Besides, this fact yields the regularity conditions needed for the integrand. In fact, the conditional expectation and the density function are continuous for $s,t\in[0,T]$ and x,y in a neighborhood of u. Hence, we can pass to the limit w.r.t. u inside the integral sign

$$\mu(J)=\iint_J\sum_{m,n=0}^{\infty}\mathbb{E}\left[Z'^{+}(s)Z'^{+}(t) \mid X(s)=X(t)=u, \nu_s=m, \nu_t=m+n\right]$$
$$p_{X(s),X(t)|\nu_s=m,\nu_t-\nu_s=n}(u,u)p_{\nu_s}(m)p_{\nu_t}(m+n)dsdt.$$

So far we have stated that $\mathbb{E}\,\mu(J)$ equals the integral in the r.h.s. of Equation (4.7) for any interval $J=J_1\times J_2\subset[0,T]^2\setminus\Delta$. As both sides represent measures on $[0,T]^2\setminus\Delta$, the result follows by the standard extension arguments of Measure Theory.

Step 2. Due to the stationarity, we can develop Formula (4.7) a little further:

$$\mathbb{E}\, U^c_{u,[2]} = 2\int_0^T (T-\tau)\sum_{n=0}^{\infty}\mathbb{E}\left[Z_0'^{+}Z_\tau'^{+} \mid Y_0(0)=Y_n(\tau)=u\right]\cdot$$
$$\cdot\, p_{Y_0(0),Y_n(\tau)}(u,u)p_{\nu_\tau}(n)d\tau,$$

where we set $Y_k(t):=Z(t)+A_k$, for $t\in[0,T]$ and $k\in\mathbb{N}$.

In effect, conditioned on $\nu_s=m, \nu_{t-s}=n$ (if $s<t$; similarly on the other case) we have $X(s)=Z(s)+A_m=Y_m(s)$ and $X(t)=Z(t)+A_{m+n}=Y_{m+n}(t)$. It is easy to see that the vector $(Y_m(s),Y_{m+n}(t))$ is independent from ν and has centered Gaussian distribution with variances 1 and covariance $r(t-s)+\rho^n/2$, in particular, this law does not depend on s,t but on their difference $t-s$, neither it depends on m. Therefore, the conditional expectation in Formula (4.7) reduces to that in the r.h.s. of the claimed formula.

Besides, for the densities functions we have

$$p_{X(s),X(t)|\nu_s,\nu_{t-s}}(u,u)p_{\nu_s}(m)p_{\nu_{t-s}}(n) = p_{Y_m(s),Y_{m+n}(t)}(u,u)p_{\nu_s}(m)p_{\nu_{t-s}}(n)$$
$$= p_{Y_0(0),Y_n(\tau)}(u,u)p_{\nu_s}(m)p_{\nu_\tau}(n).$$

There is only one factor that depends on m, and clearly $\sum_m p_{\nu_s}(m) = 1$, so the integrand is independent from m.

Finally we make the change of variables $(s,t) \mapsto (s, \tau = t-s)$, and obtain the desired inequality.

In the next two steps we bound each factor in the integrand in the formula given in Step 2.

Step 3. Assume that $\nu_\tau = n$. Let us perform the regression of $Z'(0)$ and $Z'(\tau)$ w.r.t. the event $C = \{X_0(0) = X_n(\tau) = u\}$. Therefore, we need α and β such that the random variable $Z'(0) - \alpha X_0(0) - \beta X_n(\tau)$ is independent from $X_0(0)$ and $X_n(\tau)$, analogous definition and conditions hold for $Z'(\tau)$. Routine computations show that α and β are the solutions of

$$\begin{bmatrix} 1 & r(\tau)+\rho^n/2 \\ r(\tau)+\rho^n/2 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ r'(\tau) \end{bmatrix}$$

Therefore

$$\mathbb{E}\left(Z'(0) \mid C\right) = -\mathbb{E}\left(Z'(\tau) \mid C\right) = -\frac{r'(\tau)u}{1-(r(\tau)+\rho^n/2)}.$$

and

$$var(Z'(0) \mid C) = var(Z'(\tau) \mid C) = \lambda_2 - \frac{r'(\tau)^2}{1-(r(\tau)+\rho^n/2)^2},$$

Therefore, using the well known inequalities $a^+b^+ \leq (a+b)^2/4$ and $(a+b)^2 \leq 2(a^2+b^2)$, adding and subtracting $\mathbb{E}\left(Z'(0) \mid C\right)$, it follows that

$$\mathbb{E}\left(Z'^+(0)Z'^+(\tau) \mid C\right) \leq \frac{1}{2}(var(Z'(0) \mid C) + var(Z'(\tau) \mid C))$$

Then, we have

$$\mathbb{E}\left(Z'^+(0)Z'^+(\tau) \mid C\right) \leq \lambda_2 - \frac{r'(\tau)^2}{1-(r(\tau)+\rho^n/2)^2}.$$

Now, let us assume that $n \geq 1$. Then, since $\rho < 1$, the denominator in last formula is bounded away from zero as $\tau \to 0$, in effect $1-(r(\tau)+\rho^n/2) \to 1/2 - \rho^n/2 > 0$. Therefore, it is trivially bounded above by λ_2.

To end this case, let us look at the density of the vector $(Y_0(0), Y_n(\tau))$, which has centered Gaussian distribution with variance matrix $\Sigma = \begin{pmatrix} 1 & r(\tau)+\rho^n/2 \\ r(\tau)+\rho^n/2 & 1 \end{pmatrix}$. Therefore, the denominator in the density is of the same form than in the conditional expectation and the exponential is

$$\exp\left(-\frac{1}{2|\det(\Sigma)|}\begin{pmatrix} u & u \end{pmatrix}\Sigma^{-1}\begin{pmatrix} u \\ u \end{pmatrix}\right) = \exp\left\{-\frac{u^2}{1+r(\tau)+\rho^n/2}\right\}.$$

Since $n \geq 1$ we can bound $\rho^n \leq |\rho| < 1$, hence

$$\exp\left\{-\frac{u^2}{1+r(\tau)+\rho^n/2}\right\} \leq \exp\left\{-\frac{u^2}{1+r(\tau)+|\rho|/2}\right\} = o(\varphi(u)).$$

Replacing in Inequality (4.7) of Step 1 we have that the sum starting at $n = 1$ is bounded by

$$\begin{aligned} &2T \\ &\cdot \int_0^T \sum_{n=1}^{\infty} p_{\nu_\tau}(n) \frac{\lambda_2(1-(r(\tau)+\rho^n/2)^2) - r'(\tau)^2}{(1-(r(\tau)+\rho^n/2)^2)^{3/2}} \exp\left\{-\frac{u^2}{1+r(\tau)+|\rho|/2}\right\} d\tau \\ &\qquad \leq 2T^2 \frac{\lambda_2}{\sqrt{1-((1+|\rho|)/2)^2}} \exp\left\{-\frac{u^2}{(3+|\rho|)/2}\right\} \\ &\qquad\qquad = o(\varphi(u)). \end{aligned}$$

Finally, it rest to consider the case $n = 0$, that is, when there are no jumps in $[0, T]$. Observe that the probability of $\nu_T = n = 0$ is $e^{-\lambda T}$,

We proceed as in Azaïs & Wschebor [12, Prop. 4.2], in particular, we need Geman condition to ensure the convergence of the integral. In effect, the integrand is (a constant times)

$$\frac{\lambda_2(1/2 - r^2(\tau)) - r'^2(\tau)}{(1/2 - r^2(\tau))^{3/2}} \exp\left(-\frac{u^2}{1+r(\tau)}\right).$$

Here, the second factor is bounded by $\exp(-2u^2/3) = o(\varphi(u))$.

For the first one, we have to take care on a neighborhood of zero, we use the expansion defining θ in Geman Condition (4.5). Further, we use the fact that $\theta(\tau), \theta'(\tau), \theta''(\tau) \geq 0$. To prove this note that $-r''(\tau)/\sqrt{\lambda_2}$ is the covariance function of the process $(Z'(t)/\sqrt{\lambda_2} : t \in [0, T])$, see for example Cramér & Leadbetter [23, Sc. 9.3.], hence, taking derivatives on both sides of Equation (4.6) and using Cauchy-Schwarz Inequality we deduce that

$$1 \geq -\frac{r''(\tau)}{\sqrt{\lambda_2}} = 1 - \frac{\theta''(\tau)}{\sqrt{\lambda_2}}$$

thus, $\theta''(\tau) \geq 0$ for τ in a neighborhood of *zero*, the inequalities for θ and θ' follow from this one integrating w.r.t. τ.

Therefore

$$\begin{aligned}\lambda_2(1/2 - r^2(\tau)) - r'^2(\tau) &= 2\lambda_2\tau\theta'(\tau) - \left[\frac{\lambda_2^2}{4}\tau^4 + \theta''(\tau) + 2\theta + \theta'^2\right] \\ &\leq 2\lambda_2\tau\theta'(\tau)\end{aligned}$$

and

$$1/2 - r^2(\tau) \sim \lambda_2\tau^2.$$

Therefore, the first factor is bounded by (an equivalent of) $2\tau^{-2}\theta'(\tau)$. This show that the integral is convergent.

Putting all this together, the second moment is $o(\varphi(u))$.

Claim 4.2. *We have* $\mathbb{E}\, U^d_{u,[2]} = o(\varphi(u))$.

By the arguments in Corollary 4.1 it suffices to show that

$$\mathbb{E}\, U^d_{u,[2]} \leq c\, p_{|\rho|}(u),$$

for some constant c and u large enough. We can write

$$U^d_u = \sum_{n=0}^{\nu_T}\sum_{k=0}^{n} \mathbb{I}\{X(\tau_k^-) < u, X(\tau_k) > u\},$$

hence, making the product and taking expectation, we have:

$$\begin{aligned}\mathbb{E}\, U^d_{u,[2]} &= \sum_{n=2}^{\infty}\sum_{1=k<\ell}^{n-1} p_{\nu_T}(n) \\ &\quad \cdot \mathbb{P}(Z(\tau_k) + A_{k-1} < u; Z(\tau_k) + A_k > u; Z(\tau_\ell) + A_{\ell-1} < u; Z(\tau_\ell) + A_\ell > u) \\ &\leq \sum_{n=2}^{\infty}\sum_{k<\ell=1}^{n-1} p_{\nu_T}(n)\mathbb{P}(Z(\tau_k) + A_{k-1} < u; Z(\tau_\ell) + A_\ell > u). \quad (4.8)\end{aligned}$$

Besides,

$$\begin{aligned}&\mathbb{P}(Z(\tau_k) + A_{k-1} < u; Z(\tau_\ell) + A_\ell > u) \\ &\quad = \int_0^T ds \int_s^T dt p_{\tau_k,\tau_\ell|\nu_T=n}(s,t)\mathbb{P}(Z(s) + A_{k-1} < u; Z(t) + A_\ell > u) \\ &\quad = \int_0^T ds \int_s^T dt p_{\tau_k,\tau_\ell|\nu_T=n}(s,t)\mathbb{P}(Z(0) + A_{k-1} < u; Z(t-s) + A_\ell > u),\end{aligned}$$

in the latter equality we used the stationarity of the process $\mathcal{Z}$. The vector $(Z(0)+A_{k-1}; Z(t-s)+A_\ell)$ is centered Gaussian with variances $1/2$ and covariance $r(\tau)+\rho^{\ell-k+1}/2$. It is easy to check that $-r(\tau)-\rho^{\ell-k+1}/2 < 1/2(1+|\rho|)$ which is the covariance of the vectors $Z+S$ and $Z+|\rho|S+\sqrt{1-\rho^2}V$, therefore, by the Plackett-Slepian Inequality, see Azaïs & Wschebor[12, Section 2.1], we have

$$\mathbb{P}(Z(0)+A_{k-1} < u; Z(t-s)+A_\ell > u) \leq \mathbb{P}(Z+S < u; Z+|\rho|S+\sqrt{1-\rho^2}V > u).$$

This bound does not depend on s,t, hence

$$\begin{aligned}
&\mathbb{P}(Z(\tau_k)+A_{k-1} < u; Z(\tau_\ell)+A_\ell > u) \\
&\quad \leq \mathbb{P}(Z+S < u; Z+|\rho|S+\sqrt{1-\rho^2}V > u) \int_0^T ds \int_s^T dt p_{\tau_k,\tau_\ell|\nu_T=n}(s,t) \\
&\quad = \mathbb{P}(Z+S < u; Z+|\rho|S+\sqrt{1-\rho^2}V > u) \\
&\quad = p_{|\rho|(u)},
\end{aligned}$$

since $\tau_k, \tau_\ell \mid \nu_T = n$ is concentrated on $[0,T]^2$. Finally, replacing in Equation (4.8) we have

$$\mathbb{E}\, U^d_{u,[2]} \leq \sum_{n=2}^{\infty} \sum_{1=k<\ell}^{n-1} p_{|\rho|(u)} = \frac{(\lambda T)^2}{2} p_{|\rho|(u)},$$

and the result follows.

Claim 4.3. *We have* $\mathbb{E}\, U_{u,[2]} = o(\varphi(u))$.

In effect

$$\begin{aligned}
\mathbb{E}\, U_{u,[2]} &= \mathbb{E}\,(U^c_u + U^d_u)(U^c_u + U^d_u - 1) \\
&= \mathbb{E}\, U^c_{u,[2]} + \mathbb{E}\, U^d_{u,[2]} + 2\mathbb{E}\, U^c_u U^d_u \\
&\leq \mathbb{E}\, U^c_{u,[2]} + \mathbb{E}\, U^d_{u,[2]} + 2\sqrt{\mathbb{E}\,(U^c_u)^2 \mathbb{E}\,(U^d_u)^2} \\
&= \mathbb{E}\, U^c_{u,[2]} + \mathbb{E}\, U^d_{u,[2]} + 2\sqrt{(\mathbb{E}\, U^c_{u,[2]} + \mathbb{E}\, U^c_u)\mathbb{E}\,(U^d_{u,[2]} + \mathbb{E}\, U^d_u)},
\end{aligned}$$

where we used Cauchy-Schwarz inequality. The first two terms in the r.h.s. in the last line are treated in the previous Claims, under the square root sign the first factor is equivalent to $\varphi(u)$ and the second one is $o(\varphi(u))$. Thus, $\mathbb{E}\, U_{u,[2]} = o(\varphi(u))$.

Claim 4.4. *We have* $\mathbb{P}(X(0) > u, U_u \geq 1) = o(\varphi(u))$.

Again, we follow the proof of the analogue assertion in Azaïs & Wschebor [12, Prop. 4.2].

Let us express the probability in a more convenient way

$$\begin{aligned}\mathbb{P}(X(0), U_u > 0) &\leq \mathbb{P}(X(0) > u, X(T) > u) + \mathbb{P}(X(0) > u, X(T) < u, U_u > 0)) \\ &\leq \mathbb{P}(X(0) > u, X(T) > u) + \mathbb{P}(D_u > 1)\end{aligned}$$

For the first resulting term observe that the conditions $X(0) > u, X(T) > u$ clearly imply $X(0) + X(T) > 2u$. Now we proceed as in Step 3 of Claim 4.1. For $n \geq 0$, condition on the event $\nu_T = n$. Observe that for $n \geq 1$ the vector $(X(0), X(T))$ is non-degenerated with centered Gaussian distribution with covariance matrix $\begin{pmatrix} 1 & r(0,T)+\rho^n/2 \\ r(0,T)+\rho^n/2 & 1 \end{pmatrix}$. For $n = 0$, the condition $r(0,T) \neq \pm\frac{1}{2}$ (actually $|r(0,T)| < \frac{1}{2}$) ensures the non-degeneracy of $(X(0), X(T))$, now the covariance matrix is $\begin{pmatrix} 1 & r(0,T)+1/2 \\ r(0,T)+1/2 & 1 \end{pmatrix}$.

We show that one of this terms is $o(\varphi(u))$, the others are treated analogously. Consider then the case $n = 0$. Let Y be a centered Gaussian random variable with variance a half independent from $X(0)$, thus, we can write

$$X(T) = cX(0) + \sqrt{1-c^2}Y$$

being $c = r(0,T)$, therefore $X(0)+X(T) = (1+c)X(0)+\sqrt{1-c^2}Y$ has a centered Gaussian distribution with variance $\frac{1}{2}(1+c)^2 + \frac{1}{2}(1-c^2) = 1+c$. So the desired probability can be found as

$$\begin{aligned}&\mathbb{P}(X(0) > u, X(T) > u \mid \nu_T = 0) \\ &\qquad = \int_{2u}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{1+r(0,T)}} \exp\left(-\frac{x^2}{2(1+r(0,T))}\right) dx\end{aligned}$$

The result follows since $1 + r(0,T) < 3/2$, thus, we have proved that $\mathbb{P}(X(0) > u, X(T) > u \mid \nu_T = 0) = o(\varphi(u))$.

For the last term. The key fact is that the distribution of the process $\mathcal{J}$ remains unchanged under time reversal $t \mapsto T - t$.

In effect, let us condition on the number of jumps $\nu_T = n$, then it is easy to check that the (conditional) distribution of $(A_1, A_2, \ldots, A_n) \mid \nu_T = n$ is the same as the distribution of $(A_n, A_{n-1}, \ldots, A_1) \mid \nu_T = n$, namely, the centered Gaussian distribution with covariance matrix

$$\begin{bmatrix} 1 & \rho & \rho^2 & \ldots & \rho^{n-1} \\ \rho & 1 & \rho & \ldots & \rho^{n-2} \\ \rho^2 & \rho^1 & 1 & \ldots & \rho^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \ldots & 1 \end{bmatrix}$$

Besides, it is a very well known fact that the distribution of $(\tau_1, \tau_2, \ldots, \tau_n) \mid \nu_T = n$ is that of an uniform (ordered) sample of size n, so it looks the same from 0 and from T. Since the construction of the process $\mathcal{J}$ depends on these elements and there is no difference if we start at 0 or at T the claim follows.

By a similar reasoning to the one that lead to Inequality (4.3), observe that

$$\mathbb{I}\{D_u > 1\} = \frac{1}{u} D_u (D_u - 1)$$

Hence

$$\mathbb{P}(D_u > 1) \leq \frac{1}{2} \mathbb{E}\, D_{u,[2]},$$

and by the invariance of the distribution of $\mathcal{X}$ under time reversal, we have $\mathbb{E}\, D_{u,[2]} = \mathbb{E}\, U_{u,[2]}$ and

$$\mathbb{P}(D_u > 1) \leq \frac{1}{2} \mathbb{E}\, U_{u,[2]},$$

This term has been already treated in the previous claims.

In conclusion,

$$\begin{aligned}\mathbb{P}(M(t) > u) &\geq 1 - \Phi(u) + T\sqrt{\frac{\lambda_2}{2\pi}}\,\varphi(u) + \lambda T\, p_\rho(u) + O(\varphi((1+\delta)u)) \\ &= 1 - \Phi(u) + T\sqrt{\frac{\lambda_2}{2\pi}}\,\varphi(u) + o(\varphi(u)).\end{aligned}$$

Taking into account Inequality (4.4), this completes the proof. □

4.3 Maximum of a Compound Poisson Process on a compact interval

On this section we consider the behavior of the tail of the distribution of the maximum of a pure jump process, namely, the Compound Poisson Process (CPP) with standard Gaussian jumps on a compact interval.

4.3.1 Introduction

To be more precise, we study the asymptotic probability that the maximum of the process exceeds the level u on the interval $[0, T]$ as the level u grows to infinity; in symbols

$$\mathbb{P}\left(\max_{t\in[0,T]} X(t) > u\right), \quad u \to \infty$$

We give an upper bound for this probability, the leading term of the bound is equivalent to $\mathbb{P}(X(T) > u)$.

This behavior is consistent with the structure of the covariances of such processes since the increments of the process on disjoint intervals are positive correlated.

This result is interesting, since there are well known results which intimately relate the probability that the maximum of the process exceeds certain level with the probability that the process exceeds the same level at a fixed instant. at a single point in the interval, namely, the point where the variance of the process reaches its maximum. See Berman [15] for a sharp bound for a general smooth Gaussian process or Berman [16] for an equivalence for a Gaussian processes with stationary increments and convex covariance $c(t)$ such that $c(t) = o(t)$; this include Fractional Brownian Motion. See also Albin [3] for an equivalence on the case of a Compound Poisson Process with drift or a Lévy Process with light tails and finite variation.

Consider a simple Poisson Process $(\nu_t : t \geq 0)$ as defined in Equation (4.1). Further, in order to define the Compound Poisson Process $\mathcal{X}$, let $(\xi_k : k \geq 1)$ be i.i.d. Standard Gaussian random variables, independent from $(\tau_k : k \geq 0)$. Then, the Process $\mathcal{X} = (X(t) : t \geq 0)$ is defined by

$$X(t) = \sum_{k=0}^{\nu_t} \xi_k$$

Therefore, $\mathcal{X}$ jumps the (random) quantity ξ_k at the (random) k-th jump epoch τ_k.

Theorem 4.5. *Let $\mathcal{X}$ be a CPP with standard Gaussian jumps, therefore, as $u \to \infty$ we have*

$$\mathbb{P}\left(\max_{t\in[0,T]} X(t) > u\right) \leq \Phi\left(\sqrt{2}u\right) \sum_{n=1}^{\infty} p_{\nu_T}(n) \overline{\Phi}\left(\frac{u}{\sqrt{n}}\right)$$
$$+ \overline{\Phi}(u)(1 - \Phi\left(\sqrt{2}u\right) p_{\nu_T}(1)) + \Phi\left(\sqrt{2}u\right) \sum_{n=1}^{\infty} p_{\nu_T}(n) \sum_{k=1}^{n-1} \overline{\Phi}\left(\frac{u}{\sqrt{k}}\right)$$

Moreover, the r.h.s. is equivalent to $\mathbb{P}(X(T) > u)$.

In this proof we use Hermite polynomials and Mehler Formula, see Nualart [65] and Peccati & Taqqu [67]. The reader is also referred to Chapter 6, where we use these tools quite deeply.

4.3.2 Proof

The Hermite polynomial of degree k is defined by

$$H_k(x) = (-1)^k e^{z^2/2} \left. \frac{d^k}{dt^k} e^{-t^2/2} \right|_{t=z}.$$

The family $(H_k : k \geq 0)$ is an orthogonal complete system in $L^2(\varphi(z)dz)$, see Chapter 6 for details. In addition

$$\begin{aligned} \langle H_k, H_\ell \rangle &= \int_{\mathbb{R}} H_k(z) H_\ell(z) \varphi(z) dx \\ &= \mathbb{E}\left(H_k(Z) H_\ell(Z)\right) \\ &= \delta_{k\ell} k! \end{aligned}$$

for Standard Gaussian Z. Therefore, we can compute the covariances (inner products) in terms of these polynomials. More precisely.

Mehler's Formula: Let $f, g \in L^2(\varphi(z)dz)$ and assume that $f = \sum_{k=1}^{\infty} f_k H_k$, $g = \sum_{k=1}^{\infty} g_k H_k$ and $(Z, W) \sim N(0, \Sigma)$ with $\Sigma = \left(\begin{smallmatrix} 1 & \rho \\ \rho & 1 \end{smallmatrix}\right)$, then

$$cov\left(f(Z), g(W)\right) = \sum_{k=1}^{\infty} f_k g_k k! \rho^k.$$

We begin now with the proof.

Set $S_0 = 0$ and for $k \geq 1$ let

$$S_k = \sum_{j=0}^{k} \xi_j, \quad p_k^+(u) = \mathbb{P}\left(S_{k-1} < u, S_k \geq u\right)$$

Note that, since the paths of the process $\mathcal{X}$ are piecewise constant and right-continuous, there is an up-crossing through the level u at the k-th jump epoch if and only if $S_{k-1} < u$ and $S_k \geq u$. Besides, $p_k^+(u)$ is the probability of having an up-crossing on the k-th jump epoch of $\mathcal{X}$.

Let $U_u = U_u^{\mathcal{X}}([0,T])$ be the number of up-crossings through the level u by $\mathcal{X}$ on the interval $[0,T]$. In order to get the result we use the simple inequality $\mathbb{P}(\max X(t) > u) \leq \mathbb{E}\, U_u$.

Therefore, by the nesting property of conditional expectation

$$\begin{aligned} \mathbb{E}\, U_u &= \mathbb{E}\left[\mathbb{E}\left(U_u \mid \nu_T\right)\right] \\ &= \sum_{n=1}^{\infty} \mathbb{P}(\nu_T = n) \sum_{k=1}^{n} p_k^+(u), \end{aligned} \tag{4.9}$$

where we use that the number of crossings in the n jump epochs occurring on $[0,T]$ is the sum of the indicators of the events "*there is a crossing at the k-th jump epoch*", $k=1,\dots,n$.

Now, let us compute the probabilities $p_k^+(u)$. Suppose $u>0$, then

$$p_1^+(u) = 1-\Phi(u)$$

and for $k>1$, we express $p_k^+(u)$ in terms of the covariances of the indicators

$$\begin{aligned} p_k^+(u) &= \mathbb{E}\ \mathbb{I}_{(-\infty,u)}(S_{k-1})\mathbb{I}_{[u,\infty)}(S_k) \\ &= cov(\mathbb{I}_{(-\infty,u)}(S_{k-1}),\mathbb{I}_{[u,\infty)}(S_k)) + \mathbb{E}\ \mathbb{I}_{(-\infty,u)}(S_{k-1})\mathbb{E}\ \mathbb{I}_{[u,\infty)}(S_k) \\ &= cov(\mathbb{I}_{(-\infty,u)}(S_{k-1}),\mathbb{I}_{[u,\infty)}(S_k)) + \mathbb{P}(S_{k-1}<u)\mathbb{P}(S_k\geq u). \quad (4.10) \end{aligned}$$

This fact can be verified directly from the definition of covariance. It is clear, from its definition, that S_k is a centered Gaussian random variable with variance k, thus

$$\mathbb{P}(S_{k-1}<u) = \Phi\left(\frac{u}{\sqrt{k-1}}\right), \quad \mathbb{P}(S_k\geq u) = \overline{\Phi}\left(\frac{u}{\sqrt{k}}\right)$$

where $\overline{\Phi}(z) = 1-\Phi(z)$.

The idea now is to compute these covariances using Hermite polynomials. We need the following expansion for indicator (characteristic) functions. See Slud [77] for a similar but more developed argument.

By Lemma 4.2, setting $\mathbb{I}\{(-\infty,a)\} = \sum_{k=0}^{\infty} f_k H_k$ and $\mathbb{I}\{[a,\infty)\} = \sum_{k=0}^{\infty} g_k H_k$, we have

$$f_k = -g_k = -\frac{1}{k!}H_{k-1}(a)\varphi(a).$$

Consider the standardized sums $\frac{S_{k-1}}{\sqrt{k-1}}, \frac{S_k}{\sqrt{k}}$, observe that they are jointly Gaussian random variables with means zero, variances 1 and covariance $\rho=\sqrt{(k-1)/k}$. Therefore, Mehler's Formula implies

$$\begin{aligned} p_k^+(u) &= \Phi\left(\frac{u}{\sqrt{k-1}}\right)\left[1-\Phi\left(\frac{u}{\sqrt{k}}\right)\right] \\ &- \sum_{m=1}^{\infty}\frac{1}{m!}H_{m-1}\left(\frac{u}{\sqrt{k-1}}\right)\varphi\left(\frac{u}{\sqrt{k-1}}\right)H_{m-1}\left(\frac{u}{\sqrt{k}}\right)\varphi\left(\frac{u}{\sqrt{k}}\right)\left[\frac{k-1}{k}\right]^{m/2} \\ &= \mathbb{P}\left(Z>\frac{u}{\sqrt{k-1}}, W>\frac{u}{\sqrt{k}}\right) =: \bar{F}_\rho\left(\frac{u}{\sqrt{k-1}},\frac{u}{\sqrt{k}}\right), \quad (4.11) \end{aligned}$$

where we use Lemma 4.3. Here (Z,W) a centered Gaussian vector with variances 1 and covariance $\rho=\sqrt{(k-1)/k}$.

Finally, we bound this probability in Lemma 4.4, we have

$$\bar{F}_\rho(a,b) \geq \sqrt{1+\rho}\cdot\overline{\Phi}\left(\frac{a}{\sqrt{1-\rho^2}}\right)\overline{\Phi}\left(\frac{b}{\sqrt{1+\rho}}\right)$$

Specializing for $a = u/\sqrt{k-1}$, $b = u/\sqrt{k}$ and $\rho = \sqrt{(k-1)/k}$ we have

$$\bar{F}_{\sqrt{\frac{k-1}{k}}}\left(\frac{u}{\sqrt{k-1}},\frac{u}{\sqrt{k}}\right) \geq \sqrt{1+\sqrt{\frac{k-1}{k}}}\cdot\overline{\Phi}\left(\frac{\frac{u}{\sqrt{k-1}}}{\sqrt{1-\frac{k-1}{k}}}\right)\overline{\Phi}\left(\frac{\frac{u}{\sqrt{k}}}{\sqrt{1+\sqrt{\frac{k-1}{k}}}}\right)$$

Consequently, for $k > 1$

$$p_k^+(u) \leq \Phi\left(\frac{u}{\sqrt{k-1}}\right)\overline{\Phi}\left(\frac{u}{\sqrt{k}}\right) - \sqrt{1+\sqrt{\frac{k-1}{k}}}\cdot\overline{\Phi}\left(\frac{\sqrt{k}u}{\sqrt{k-1}}\right)\overline{\Phi}\left(\frac{u}{\sqrt{k}\sqrt{1+\sqrt{\frac{k-1}{k}}}}\right)$$

Now, $\Phi(\cdot) \leq 1$, $\sqrt{k/(k-1)} \leq \sqrt{2}$, hence

$$\bar{\Phi}\left(\sqrt{\frac{k}{k-1}}u\right) \geq \bar{\Phi}\left(\sqrt{2}u\right)$$

and

$$\sqrt{2} \geq \sqrt{1+\sqrt{\frac{k-1}{k}}} \geq 1 \Rightarrow \bar{\Phi}\left(\frac{u}{\sqrt{k}\sqrt{1+\sqrt{\frac{k-1}{k}}}}\right) \geq \bar{\Phi}\left(\frac{u}{\sqrt{k}}\right)$$

Therefore

$$p_k^+(u) \leq \overline{\Phi}\left(\frac{u}{\sqrt{k}}\right) - \overline{\Phi}\left(\sqrt{2}u\right)\overline{\Phi}\left(\frac{u}{\sqrt{k}}\right)$$

Putting this together with Equation (4.9) for the expectation, we obtain the

desired result

$$\begin{aligned}\mathbb{E}\,U_u &\le \overline{\Phi}(u) + \Phi\left(\sqrt{2}u\right)\sum_{n=2}^{\infty} p_{\nu_T}(n)\sum_{k=2}^{n}\overline{\Phi}\left(\frac{u}{\sqrt{k}}\right)\\ &= \Phi\left(\sqrt{2}u\right)\sum_{n=1}^{\infty} p_{\nu_T}(n)\overline{\Phi}\left(\frac{u}{\sqrt{n}}\right)\\ &+ \overline{\Phi}(u)(1-\Phi\left(\sqrt{2}u\right)p_{\nu_T}(1)) + \Phi\left(\sqrt{2}u\right)\sum_{n=1}^{\infty} p_{\nu_T}(n)\sum_{k=1}^{n-1}\overline{\Phi}\left(\frac{u}{\sqrt{k}}\right)\end{aligned}$$

Let us see that this result is in tune with Berman's results. Note that the CPP is a diffusive process, in particular the variance achieves its maximum value at the point $t = T$. Actually, the first term in the r.h.s. is the leading term and is equivalent to the tail of the distribution of $X(T)$. In effect, since $\Phi(\sqrt{2}u) \to 1$ as $u \to \infty$, conditioning on $\nu_T = n$ it is immediate that the first term is equivalent to $\mathbb{P}(X(t) > u)$. To see that the first term is the leading one, just note that if $a > b$, then $\overline{\Phi(u/b)}/\overline{\Phi(u/b)} \to 0$ as $u \to \infty$.

We end the section with some auxiliary results.

Lemma 4.2. *Let $f = \mathbb{I}_{(-\infty,a]}$ and $g = \mathbb{I}_{[a,\infty)}$, Then, (after centering) the Hermite coefficients f_k and g_k of f and g are given by $f_0 = g_0 = 0$ and for $k \ge 1$*

$$\begin{aligned} f_k &= -\frac{1}{k!}H_{k-1}(a)\varphi(a)\\ g_k &= -f_k.\end{aligned}$$

Proof. Consider a Standard Gaussian random variable Z. In order to apply Mehler's Formula we need to center the random variables, thus, let $m = \mathbb{E}\,f(Z) = \mathbb{E}\,\mathbb{I}_{\{Z\le a\}} = \Phi(a)$. For $k \ge 1$, we have

$$\begin{aligned} f_k = \frac{1}{k!}\mathbb{E}\left((f(Z)-m)H_k(Z)\right) &= \frac{1}{k!}\left[\int_{-\infty}^{a} H_k(z)\varphi(z)dz - m\int_{-\infty}^{\infty} H_k(z)\varphi(z)dz\right]\\ &= \frac{1}{k!}\frac{(-1)^k}{\sqrt{2\pi}}\int_{-\infty}^{a}\frac{d^k}{dz^k}e^{-z^2/2}dz - m\mathbb{E}\,H_k(Z)\\ &= -\frac{1}{k!}\cdot\frac{(-1)^{k-1}}{\sqrt{2\pi}}\left[\frac{d^{k-1}}{dz^{k-1}}e^{-z^2/2}\right]_{-\infty}^{a}\\ &= -\frac{1}{k!}H_{k-1}(a)\varphi(a),\end{aligned}$$

where we use that $\mathbb{E}\,H_k(Z) = 0$.

Now, observe that centering g we obtain

$$\mathbb{I}_{\{Z\ge a\}} - (1-\Phi(a)) = -([1-\mathbb{I}_{\{Z\ge a\}}] - \Phi(a)) = -(f-m)$$

Thus $g_k = -f_k$.

Finally, since $H_0(z) = 1$ for all z, $f_0 = \mathbb{E}\,(f(Z) - m)H_0(Z) = \mathbb{E}\,(f(Z) - m) = 0$. □

Lemma 4.3. *Let us prove the last equality in Equation* (4.11).

Proof. Let φ_ρ be the density function of the random vector (Z, W) of the thesis and $g_\rho(z, w) = \varphi_\rho(z, w)/\varphi(z)\varphi(w)$. Consider the Hermite polynomial expansion $g_\rho(z, w) = \sum_{k,m} a_{(k,m)} H_k(z) H_m(w)$, with

$$\begin{aligned} a_{(k,m)} &= \frac{1}{k!m!} \iint_{\mathbb{R}^2} g_\rho(z, w) H_k(z) H_m(w) \varphi(z)\varphi(w) dz dw \\ &= \frac{1}{k!m!} \iint_{\mathbb{R}^2} \varphi_\rho(z, w) H_k(z) H_m(w) dz dw \\ &= \frac{1}{k!m!} \mathbb{E}\,[H_k(Z) H_m(W)], \end{aligned}$$

being (Z, W) as above. By Mehler's Formula the expectation yields $\delta_{km} k! \rho^k$. Hence

$$a_{(k,m)} = \frac{1}{k!^2} \rho^k \delta_{km}.$$

Therefore, the expansion of g_ρ is given by

$$g_\rho(z, w) = \sum_{k=0}^{\infty} \frac{1}{k!} \rho^k H_k(z) H_m(w),$$

Thus

$$\varphi_\rho(z, w) = \varphi(z)\varphi(w) \sum_{k=0}^{\infty} \frac{1}{k!} \rho^k H_k(z) H_m(w),$$

Now, integrating this

$$\begin{aligned} \mathbb{P}(Z > a, W > b) &= \int_a^\infty \int_b^\infty \varphi_\rho(z, w) dz dw \\ &= \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{\rho^k}{k!} \int_a^\infty \frac{\partial^k}{\partial z^k} e^{-z^2/2} dz \int_b^\infty \frac{\partial^k}{\partial w^k} e^{-w^2/2} dw \\ &= \sum_{k=1}^{\infty} \frac{\rho^k}{k!} H_{k-1}(a)\varphi(a) H_{k-1}(b)\varphi(b). \end{aligned}$$

Finally, specializing for $a = u/\sqrt{k-1}$ and $b = u/\sqrt{k}$ the result follows. □

Lemma 4.4.

$$\bar{F}_\rho(a,b) \geq \sqrt{1+\rho}\cdot\overline{\Phi}\left(\frac{a}{\sqrt{1-\rho^2}}\right)\overline{\Phi}\left(\frac{b}{\sqrt{1+\rho}}\right)$$

Proof. The exponent in the integrand is $-\frac{1}{2(1-\rho^2)}(z^2-2\rho zw+w^2)$, completing the square we have

$$\begin{aligned}\bar{F}_\rho(a,b) &= \int_b^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2(1-\rho^2)}(1-\rho)w^2}dw \int_a^\infty \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(z-\rho w)^2}dz \\ &= \int_b^\infty \overline{\Phi}\left(\frac{a-\rho w}{\sqrt{1-\rho^2}}\right)\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2(1+\rho)}w^2}dw \\ &\geq \sqrt{1+\rho}\cdot\overline{\Phi}\left(\frac{a-\rho b}{\sqrt{1-\rho^2}}\right)\overline{\Phi}\left(\frac{b}{\sqrt{1+\rho}}\right) \\ &> \sqrt{1+\rho}\cdot\overline{\Phi}\left(\frac{a}{\sqrt{1-\rho^2}}\right)\overline{\Phi}\left(\frac{b}{\sqrt{1+\rho}}\right).\end{aligned}$$

Where we use that $\overline{\Phi}(t)$ decays when t grows. □

Part II

Random Polynomials

In this part of the thesis we apply classical Rice Formulas to some problems involving random polynomials including the number of zeros of the classical trigonometric ensemble of random polynomials and the number of roots of a square system of complex polynomial equations.

Part II is organized as follows.

In Chapter 5, we prove the conjecture that the variance of the number of roots of the Classical trigonometric polynomials is asymptotically equivalent to a constant times K as $K \to \infty$. Besides, we state a Central Limit Theorem for this number of roots.

See Dalmao & León [26].

In Chapter 6, we work on Bézout's Theorem, which is a generalization of the Fundamental Theorem of Algebra. We prove that for the case quadratic systems of size m the number of roots is almost surely equal to 2^m, and make some progress in the general case.

Chapter 5

CLT for Random Trigonometric Polynomials

In this chapter we begin our study of the number of roots of random ensembles of polynomials. More precisely, we study the asymptotic mean and variance and a Central Limit Theorem (CLT) for the number of roots of Classical Trigonometric Polynomials.

In addition to Rice Formulas, the main tool is Wiener - Hermite expansion.

5.1 Introduction and Main Result

Consider the classical random trigonometric polynomials, that is, the polynomials defined, for $K = 1, 2, \ldots$, by

$$T_K(t) := \frac{1}{\sqrt{K}} \sum_{n=1}^{K} a_n \cos(nt),$$

where the coefficients a_n are i.i.d. standard Gaussian random variables and $t \in [0, \pi]$.

This ensemble of random polynomials appears frequently in Physics, for instance in Nuclear Physics (Random Matrix Theory), Statistical Mechanics, Quantum Mechanics, Theory of Noise, see Granville & Wigman [37] and references therein.

One of the main questions about random polynomials concerns the random variable: *number of zeros* (level crossings in general); in our case the number of zeros of T_K on the interval $[0, \pi]$ (or on some other interval).

The distribution of this random variable remains unknown, and one way to

approach it is to describe it not directly but through its moments. Let us comment some of the known results about the distribution of the number of roots of Classical Trigonometric Polynomials and related ensembles of random polynomials.

The number of zeros of different ensembles of random polynomials have attracted attention of physicists and mathematicians for at least seventy years. Consequently, there is an extensive literature on the subject, starting with Littlewood & Offord [58, 57, 56] who studied algebraic polynomials, that is, polynomials of the form

$$p(t) = a_0 + a_1 t + a_2 t^2 + \dots a_K t^K,$$

with random coefficients $a_0, a_1, \dots, a_K$; these works were complemented by those of Erdös & Offord [30], Kac [44] and Ibragimov & Maslova [39, 40]. The final result for the mean number of real roots of an algebraic polynomial of degree K, is that, for *i.i.d.* coefficients in the domain of attraction of the Gaussian law, the mean number of roots is equivalent to $2\log(K)/\pi$ for centered a_n and half this quantity for non-centered a_n. Maslova [63, 62] established the asymptotic variance, namely $\frac{4}{\pi}\left(1-\frac{2}{\pi}\right)\log(K)$, and a CLT for the number of real roots of algebraic polynomials.

Classical trigonometric polynomials are intimately related with another ensemble of random trigonometric polynomials defined by

$$X_K(t) = \frac{1}{\sqrt{K}} \sum_{n=1}^{K} \left(a_n \cos(nt) + b_n \sin(nt)\right),$$

where a_n, b_n are i.i.d. standard Gaussian random variables. The polynomials X_K have the great advantage of being stationary with respect to t. This fact (and the Gaussianity) simplifies largely the treatment of the level crossing counting problem.

For this ensemble of (stationary) trigonometric polynomials Granville & Wigman [37] gave a proof of the CLT for the number of zeros, using conditions on moments of order higher than the second. After that, Azaïs & León [9] extended this result to all levels and gave a simplified proof of the CLT lying on the Wiener Chaos decomposition and Taqqu-Peccati's method. In particular, Azaïs & León [9] avoid conditions on higher moments than the second. Further, they show that the asymptotic value of the variance differs whether the level is zero or non-zero.

In order to prove this CLT, they based on the well known fact that the number of zeros, $N_X = N_X([0, K\pi])$, of the stationary, centered, Gaussian process X whose covariance function is $\mathbb{E}\, X(s)X(t) = \sin(t-s)/(t-s)$, verifies the following CLT.

Theorem 5.1. *Let X be the stationary centered Gaussian process with covariance function $c(\tau) = \sin(\tau)/\tau$. Then, the (standardized) number of zeros $N_X([0, K\pi])$ converges in distribution towards a standard Gaussian random variable, more precisely*

$$\frac{N_X[0, K\pi] - \mathbb{E}\, N_X[0, K\pi]}{\sqrt{K\pi}} \Rightarrow N(0, V^2)$$

with $0 < V < \infty$, see Azaïs & León [9].

The proof of such a result relies on the approximation of the covariance function c by compactly supported ones, more precisely, such an approximation is obtained by convolution with a compactly supported kernel. This yields an M-dependent stationary Gaussian process for which a CLT can be obtained. See Diananda [28], Malevich [60], Cuzick [25].

In Azaïs & León [9] it is shown that this result implies a CLT for the number of level crossings of the stationary trigonometric ensemble Y_K.

For Classical Trigonometric Polynomials, the asymptotic expectation of N_{T_K} on $[0, 2\pi]$ is known since Dunnage's work [29] to be $2K/\sqrt{3}$. Later, Wilkins [79] proved that the error in the approximation of the asymptotic expectation of the number of zeros is $O(1)$, actually he approximates up to the third order

$$\mathbb{E}\, N_{T_K}[0, 2\pi] = \frac{1}{\sqrt{3}} \left[(2n+1) + D_1 + \frac{D_2}{2n+1} + \frac{D_3}{(2n+1)^2} \right] + O\left(\frac{D_3}{(2n+1)^3} \right)$$

with $D_1 = 0.23\ldots$, etc.

The leading asymptotic term was proven to be the same for non-centered coefficients and for dependent coefficients with constant correlation, see Farahmand [31] and references therein or see Bharucha-Reid & Sambandham [17] for a review.

Recently, Farahmand & Li [32] considered the mean number of roots of T_K and X_K allowing its coefficients to have different means and variances, but being independent and Gaussian.

The variance was conjectured by Farahmand [31], Farahmand & Sambandham [33] and Granville & Wigman [37] to be equivalent to VK, as K grows to infinity, for some positive constant V, but was not proven to be so.

Finally, let us say that Classical trigonometric polynomials are eigenfunctions of the Laplace operator with periodic conditions. Thus, one way to generalize them to higher dimension is to consider Laplace eigenfunctions. Recently, Krishnapur, Kurlberg & Wigman [52] studied the length of the zero level curves (nodal domains) on the 2-torus of random Laplace eigenfunctions when the eigenvalue (energy level) grows to infinity.

In this chapter, we study the asymptotic behavior, as K tends to infinity, of the variance of the number of zeros of T_K on $[0,\pi]$ and prove the conjecture that the asymptotic main term of the variance is VK ($V>0$). Furthermore, we establish a Central Limit Theorem (CLT) for the number of zeros of T_K. Our main result is the following.

Denote the number of zeros of T_K on the interval $[0,\pi]$ by $N_{T_K} = N_{T_K}[0,\pi]$.

Theorem 5.2. *The normalized number of zeros of T_K on the interval $[0,\pi]$ converges in distribution to a Gaussian random variable, more precisely*

$$\frac{N_{T_K}[0,\pi] - \mathbb{E}\, N_{T_K}[0,\pi]}{\sqrt{\pi K}} \Rightarrow N(0,V^2),$$

where $0<V<\infty$ is given in Lemma 5.4.

5.2 Preliminaries

In the following lines we briefly describe the main tools that we use in the proof of Theorem 5.2.

Stochastic Integration: See Peccati & Taqqu [67] and Hiu-Hsiung Kuo [53] for details, see also Øksendal [66].

Consider a standard Brownian Motion (or Wiener Process) $B=(B_\lambda : \lambda\in[0,1])$. Denote thee Lebesgue measure on $\mathbb{R}^d$ ($d\geq 1$) by $d\lambda$, and the Borel σ-algebra by $\mathcal{B}$. The stochastic integral with respect to B is defined on $L^2([0,1]) = L^2([0,1],\mathcal{B},d\lambda)$, as follows: for a simple function

$$h=\sum_{k=1}^{n} a_k \mathbb{I}A_k,$$

where $n\in\mathbb{N}$, $a_k\in\mathbb{R}$ and the $A_k's$ are pairwise disjoint intervals in $[0,1]$; we set

$$I_1^B(h) = B(h) = \int_0^1 h(\lambda)dB_\lambda := \sum_{k=1}^{n} a_k \Delta B(A_k),$$

being $\Delta B(\lambda_1,\lambda_2) = B_{\lambda_2} - B_{\lambda_1}$. One can verify directly that

$$\mathbb{E}\, I_1^B(g) I_1^B(h) = \int_0^1 g(\lambda)h(\lambda)d\lambda.$$

Using this isometric property and the fact that the simple functions are dense in $L^2([0,1])$, one can extend the definition of the stochastic integral with respect to

B to every function in $L^2([0,1])$. Furthermore, the isometric property remains valid in $L^2([0,1])$.

In order to define the q-fold multiple stochastic integral I_q^B with respect to B, it is convenient to remove the diagonals. This is Itô's idea and it results in orthogonal random variables $I_q^B(\cdot)$ for different values of q, on the other hand, if we do not remove diagonals, Wiener's initial idea, the resulting integrals are not orthogonal for different values of q. We follow Itô's method.

The construction of the integral is done analogously to the unidimensional case, but using indicator functions of rectangles whose "sides" are pairwise disjoint.

The key relations that we need are: the **isometric property**, see Peccati & Taqqu [67, Eq. 5.5.62],

$$\mathbb{E}\, I_p^B(g_p) I_q^B(h_q) = \delta_{pq} q! \int_{[0,1]^q} g_p(\lambda) h_q(\lambda) d\lambda. \tag{5.1}$$

with δ the Kronecker's delta, and the **multiplication formula**, see Peccati & Taqqu [67, Eq. 6.4.17],

$$I_p^B(g_p) I_q^B(h_q) = \sum_{n=0}^{p\wedge q} n! \binom{p}{n} \binom{q}{n} I_{p+q-2n}^B(g_p \otimes_n h_q), \tag{5.2}$$

where $g_p \otimes_n h_q$ is the contraction of g_p and h_q defined by

$$\begin{aligned} &g_p \otimes_n h_q(x_1, \dots, x_{p+q-2n}) \\ &= \int_{[0,1]^n} g_p(z_1, \dots, z_n, x_1, \dots, x_{p-n}) h_q(z_1, \dots, z_n, x_{p-n+1}, \dots, x_{p+q-2n}) dz_1 \dots dz_n. \end{aligned} \tag{5.3}$$

Remark 5.1. *Assume that g_p and h_q are tensor products, $g_p = g^{\otimes p}$ and $h_q = h^{\otimes q}$. That is,*

$$\begin{aligned} g^{\otimes p}(z_1, \dots, z_p) &= g(z_1) \dots g(z_p), \\ h^{\otimes p}(z_1, \dots, z_q) &= h(z_1) \dots h(z_q). \end{aligned}$$

Then, if g and h are orthogonal, that is, $\int_0^1 g(\lambda) h(\lambda) d\lambda = 0$, then $g_p \otimes_n h_q$ vanish for all $n \geq 1$. Consequently the multiplication formula (5.2) simplifies to

$$I_p^B(g^{\otimes p}) I_q^B(h^{\otimes q}) = I_{p+q}^B(g^{\otimes p} \otimes h^{\otimes q}). \tag{5.4}$$

Thus in this case, the product of two stochastic integrals is a stochastic integral (rather than a combination of them).

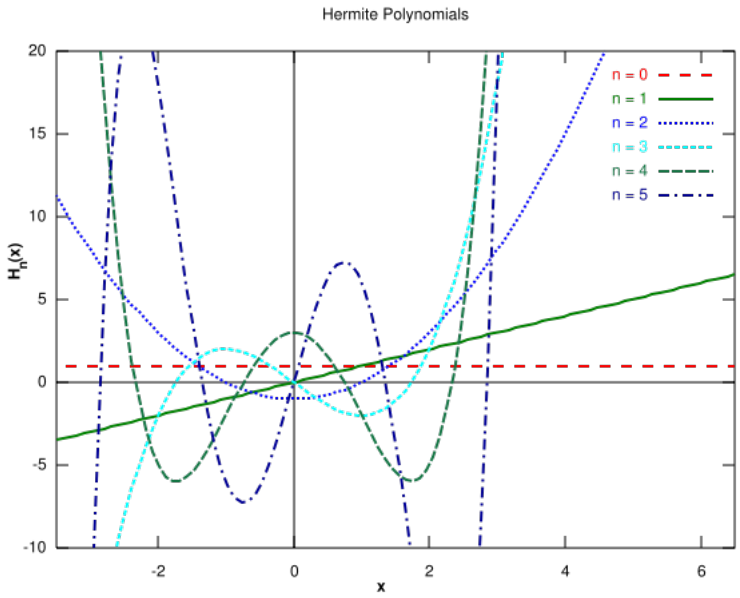

Figure 5.1: Hermite polynomials $H_q : q \leq 5$.

Hermite Polynomials and Wiener Chaos: The Hermite polynomial of degree q is defined by

$$H_q(x) = (-1)^q e^{x^2/2} \left. \frac{d^q}{dt^q} \right|_{t=x} e^{-t^2/2}.$$

For instance,

$$H_0(x) = 1, \quad H_1(x) = x, \quad H_2(x) = x^2 - 1, \quad H_3(x) = x^3 - 3x,$$

see Figure 5.2.

Alternatively, Hermite polynomials can be defined as the solutions of the equation

$$\exp\left(\theta x - \frac{\theta^2}{2}\right) = \sum_{q=0}^{\infty} \frac{\theta^q}{q!} H_q(x) \tag{5.5}$$

for all $\theta, x \in \mathbb{R}$.

Remark 5.2. *One has to be careful, since some authors, for instance Nualart [65], define H_q with other normalization constants.*

A direct computation shows that for standard Gaussian X,

$$\mathbb{E}\, H_p(X) H_q(X) = \delta_{pq} q!.$$

Moreover, the sequence $\big((q!)^{-1/2}H_q : q \geq 0\big)$ is an orthonormal basis of $L^2(\mathbb{R}, \varphi(dx))$, where, as usual, φ stands for the standard Gaussian density function.

Now, consider a (standard) Brownian Motion B_λ on the interval $[0,1]$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and stochastic integrals as above. Thus, the isometric property (5.1) implies that the map

$$h \mapsto B(h) = \int_0^1 h(\lambda) dB_\lambda,$$

defines an isometry between $L^2([0,1])$ and $L^2(\Omega, \sigma(B), \mathbb{P})$, where $\sigma(B)$ is the filtration defined by the Brownian Motion B. The process B is called an isonormal process, this definition can be extended to other processes trivially, see Peccati & Taqqu [67, Chapter 8].

From Equation (5.5) and the properties of multiple stochastic integration it can be proven that: for $h \in L^2([0,1])$ with unit norm, we have

$$I_q^B(h^{\otimes q}) = H_q(B(h)), \tag{5.6}$$

That is, the Hermite polynomial of degree q applied to the (simple) stochastic integral of h with respect to B equals the multiple stochastic integral of the tensor product, $h^{\otimes q}$, of h with itself q times.

Combining Equations (5.5) and (5.6) it follows the fundamental Hermite expansion of any square integrable functional of the Brownian motion B, that is, if $F \in L^2(\Omega, \sigma(B), \mathbb{P})$, then there exists a unique sequence of symmetric functionals $(f_q : q \geq 1)$ with $f_q \in L^2([0,1]^q, d\lambda)$, such that

$$F - \mathbb{E}\, F = \sum_{q=1}^{\infty} I_q^B(f_q), \tag{5.7}$$

where the equality holds in the L^2 sense.

For $q \geq 1$, the closed subspace of $L^2(\Omega, \sigma(B), \mathbb{P})$ spanned by the collection of all random variables of the form $I_q^B(f)$ for symmetric square-integrable $f \in [0,1]^q$ is called the q-th Wiener Chaos associated with B and denoted by $\mathcal{H}_q$, in addition we define $\mathcal{H}_0$ as the space of constants. Observe that the isometric property (5.1) implies the orthogonality of the Wiener Chaos for different values of q. Furthermore, Equation (5.7) show that

$$L^2(\Omega, \sigma(B), \mathbb{P}) = \bigoplus_{q=0}^{\infty} \mathcal{H}_q,$$

where $\oplus$ indicates an orthogonal sum.

Besides, we say that a square integrable functional has Hermite rank τ if its components in the Chaos $\mathcal{H}_q$ vanish for $q < \tau$.

Mehler Formula: Mehler Formula enable us to compute the expectation of the product of four Hermite polynomials. It is a particular case of the Diagram Formula.

Let (X_1, X_2, X_3, X_4) be a centered Gaussian vector with variance matrix

$$\begin{bmatrix} 1 & 0 & \rho_{13} & \rho_{14} \\ 0 & 1 & \rho_{23} & \rho_{24} \\ \rho_{13} & \rho_{23} & 1 & 0 \\ \rho_{14} & \rho_{24} & 0 & 1 \end{bmatrix}$$

Then, if $r_1 + r_2 = r_3 + r_4$:

$$\mathbb{E}\left[\prod_{k=1}^{4} H_{r_k}(X_k)\right] = \sum_{(d_1,d_2,d_3,d_4)\in J} \frac{r_1!r_2!r_3!r_4!}{d_1!d_2!d_3!d_4!}\rho_{13}^{d_1}\rho_{14}^{d_2}\rho_{23}^{d_3}\rho_{24}^{d_4},$$

where J is the set of indexes $d_i \geq 0$ verifying

$$d_1 + d_2 = r_1; \quad d_3 + d_4 = r_2; \quad d_1 + d_3 = r_3; \quad d_2 + d_4 = r_4.$$

If $r_1 + r_2 \neq r_3 + r_4$, then the expectation vanish.

Arcones inequality: Arcones Inequality will help us with the covariances of the terms involved in Wiener Itô expansions. See Arcones [5] and also Bardet & Surgailis [14].

Let $X = (X^{(1)}, \ldots, X^{(d)})$ and $Y = (Y^{(1)}, \ldots, Y^{(d)})$ be two centered Gaussian random vectors in $\mathbb{R}^d$. Assume that

$$\mathbb{E}\, X^{(j)}X^{(k)} = \mathbb{E}\, Y^{(j)}Y^{(k)} = \delta_{jk}$$

for all $1 \leq j, k \leq d$. Define

$$r^{(j,k)} = \mathbb{E}\, X^{(j)}Y^{(k)}.$$

Let f be a function on $\mathbb{R}^d$ with finite second moment and rank τ $(1 \leq \tau < \infty)$ with respect to X. Suppose that

$$\psi := \max\left\{\max_{1\leq j\leq d}\sum_{k=1}^{d}|r^{(j,k)}|, \max_{1\leq k\leq d}\sum_{k=1}^{d}|r^{(j,k)}|\right\} \leq 1$$

then

$$\left|\mathbb{E}\,(f(X) - \mathbb{E}\, f(X))(f(Y) - \mathbb{E}\, f(Y))\right| \leq \psi^\tau \mathbb{E}\, f^2(X).$$

Note that the sums in the definition of Arcones coefficient ψ are operator norms of the matrix of covariances (1 and ∞ norm respectively). Clearly, since all the norms in a finite dimensional linear space are equivalent, one can choose other norm with obvious changes in the formula.

Peccati-Tudor's Theorems: On this item we recall two key theorems, see Peccati & Tudor [68] and Peccati & Taqqu [67].

The first one involve sequences of chaotic variables.

Theorem 5.3. *Fix an integer $q \geq 2$. For any sequence $(f^{(k)})_k$ of symmetric functions in $L^2([0,1]^q)$ such that*

$$\lim_{k\to\infty} q!\|f^{(k)}\|_2^2 = \lim_{k\to\infty} \mathbb{E}\left[I_q^B\left((f^{(k)})^2\right)\right] = 1,$$

the following conditions are equivalent:

1. *for every $n = 1, \ldots, q-1$:*

$$\lim_{k\to\infty} \|f^{(k)} \otimes_n f^{(k)}\|_2 = 0,$$

   where $\otimes_n$ denotes the contraction as defined in Equation (5.3).

2. *as $k \to \infty$, the sequence $\left(I_q^B\left(f^{(k)}\right)\right)_k$ converges in distribution towards a standard Gaussian random variable.*

Actually, another condition, involving cumulants, is given, but the cited one are sufficient for our purposes.

The second theorem involve sequences of chaotic vectors.

Theorem 5.4. *Assume that for $q_1 \leq q_2 \leq \cdots \leq q_m$, we have functionals $f_j^{(k)}$ such that $\mathbb{E}\left[I_{q_j}(f_j^{(k)})\right]^2 \to_k \sigma_{d_{jj}}^2$, then, if $\mathbb{E}\left[I_{q_j}(f_j^{(k)}))I_{q_\ell}(f_\ell^{(k)})\right] \to_k 0$ we have*

$$(I_{q_1}(f_1^{(k)}), \ldots, I_{q_m}(f_m^{(k)})) \Rightarrow_k N(0, D_m),$$

if and only if $I_{q_j}(f_j^{(k)})$ converges in law towards a centered Gaussian random variable with variance d_{jj}. Here D_m stands for a diagonal matrix with entries d_{jj}.

One of the main features of this methods is that it does not need conditions on moments higher than the second.

5.3 Proof of the CLT

In order to obtain our results we make use of the techniques of Wiener Chaos expansion, Peccati-Tudor's method for obtaining CLTs and, of course, Rice Formulas.

More precisely, we obtain the Wiener Chaos expansion for the normalized number of zeros of the normalized version of T_K on the interval $[0, \pi]$. Then, we use Rice Formulas to bound the second factorial moment (and, thus, the variance) of the number of zero up-crossings in order to pass to the limit under the integral sign and obtain the asymptotic variance.

A key fact, is that removing the extremes of the interval, the behavior of (the covariance of the standardized version of) T_K is asymptotically equivalent to that of (the covariance of) its stationary counterpart Y_K, so the limit variances of the respective number of zeros coincide. Our work closely follows Azaïs & León [9], but the asymptotic Gaussianity is obtained through contractions rather than through the L^2 proximity to the limit process.

5.3.1 Scale change

Let us replace t by t/K, this permits us to look at the polynomials T_K at a convenient scale and to find a limit for them.

Thus, from now on we are concerned with the zeros on the interval $[0, K\pi]$ of the polynomials

$$T_K(t) = \frac{1}{\sqrt{K}} \sum_{n=1}^{K} a_n \cos\left(\frac{n}{K}t\right).$$

Let Y_K be the stationary counterpart of T_K, that is

$$Y_K(t) := X_K\left(\frac{t}{K}\right) = \frac{1}{\sqrt{K}} \sum_{n=1}^{K} \left[a_n \cos\left(\frac{n}{K}t\right) + b_n \sin\left(\frac{n}{K}t\right)\right]$$

where the coefficients a_n and b_n are independent and identically distributed standard Gaussian random variables. Clearly T_K and Y_K are centered Gaussian processes in t.

Denote by c_K the covariance function of Y_K, it is well known, see Azaïs & León [9, Eq. 1], Granville & Wigman[37, Eq. 13], etc, that

$$c_K(t) = \frac{1}{K} \sum_{n=1}^{K} \cos\left(\frac{n}{K}t\right). \tag{5.8}$$

Further, c_K can be expressed in closed form using Fejér Kernel, namely

$$c_K(t) = \frac{1}{K}\cos\left(\frac{K+1}{2K}t\right)\frac{\sin(t/2)}{\sin(t/2K)}$$

A direct computation, shows that the covariance function of the classical trigonometric polynomials, $r_K(s,t)$, is given by

$$r_K(s,t) = \frac{1}{2}(c_K(t-s) + c_K(t+s)).$$

Indeed,

$$\begin{aligned} r_K(s,t) = \mathbb{E}\, T_K(s)T_K(t) = \mathbb{E}\,\frac{1}{\sqrt{K}}\sum_{n=1}^{K} a_n \cos\left(\frac{n}{K}s\right)\cdot\frac{1}{\sqrt{K}}\sum_{m=1}^{K} a_m\cos\left(\frac{m}{K}t\right) \\ = \frac{1}{K}\sum_{n=1}^{K}\cos\left(\frac{n}{K}s\right)\cos\left(\frac{n}{K}t\right), \end{aligned}$$

where we use that the a_n are i.i.d. standard Gaussian and then $\mathbb{E}\, a_n a_m = \delta_{nm}$, δ denotes Kronecker's delta. Now, using the well known trigonometric formula $\cos(a)+\cos(b) = 2\cos((a+b)/2)\cos((a-b)/2)$ for $(a+b)/2 = ns/K$ and $(a-b)/2 = nt/K$ we obtain

$$\begin{aligned} r_K(s,t) &= \frac{1}{2K}\sum_{n=1}^{K}\left[\cos\left(\frac{n}{K}(t-s)\right) + \cos\left(\frac{n}{K}(t+s)\right)\right] \\ &= \frac{1}{2}(c_K(t-s) + c_K(t+s)) \end{aligned} \tag{5.9}$$

as claimed.

This fact shows that T_K is not stationary with respect to t, but its covariance is half of the covariance of Y_K plus a term that goes to zero as s or/and t grows to infinity. This is the key point, and it suggest that the number of zeros of these processes have the same limit behavior.

In particular, the variance of $T_K(t)$ is

$$V_K^2(t) := r_K(t,t) = \frac{1}{2}(1 + c_K(2t)). \tag{5.10}$$

Thus, the limit variance, as $t \to \infty$, is $\frac{1}{2}$ and not 1 as in the stationary case. At $t = 0$, T_K and Y_K have the same variance.

Remark 5.3. *Needless to say that the processes obtained from the T_K's replacing the cosines by sines have also asymptotic variance $\frac{1}{2}$. Furthermore, these sines-processes have covariance functions $(c_K(t-s) - c_K(t+s))/2$, thus for large values of $t+s$ the behavior of sines and cosines processes are very similar.*

From now on, we work on the case $s < t$ and denote $\tau := t - s$ and $\sigma := t + s$. The case $s > t$ is analogous. It is convenient to use the standardized version of T_K, namely

$$\overline{T}_K(t) := \frac{T_K(t)}{V_K(t)}, \tag{5.11}$$

thus $\overline{T}_K$ has unit variances and its covariance, $\overline{r}_K$, is given by

$$\overline{r}_K(s,t) = \frac{c_K(\tau) + c_K(\sigma)}{\sqrt{1 + c_K(2s)}\sqrt{1 + c_K(2t)}}.$$

Remark 5.4. *It is clear that the random variables $N^{T_K}([0,\pi])$ and $N^{\overline{T}_K}([0,\pi])$ coincide.*

5.3.2 Limit covariances

The r.h.s. in the first line of Equations (5.9), expresses r_K as a Riemann sum with partition $\{n/K : n = 0, 1, \ldots, K\}$ (s, t enter as parameters), thus, it follows that

$$r_K(s,t) \to r(s,t) := \frac{1}{2}(\mathrm{sc}(\tau) + \mathrm{sc}(\sigma)), \tag{5.12}$$

as $K \to \infty$, where sc is the cardinal sine function, that is $\mathrm{sc}(x) = \sin(x)/x$.

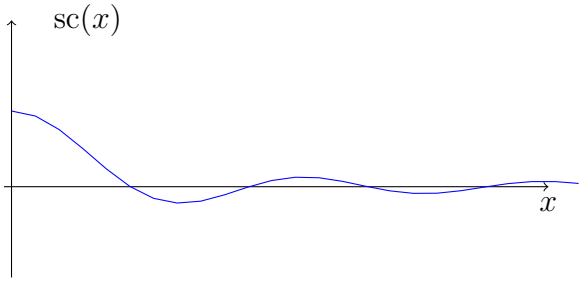

Figure 5.2: Cardinal Sine

Besides, from the boundedness of the limit function, it is easy to see that this convergence is uniform on off-diagonal compacts (compacts in $[0, K\pi]^2$ not containing points in the diagonal). Furthermore, the order one and order two derivatives of r_K converge in the same manner to the corresponding derivatives of r and the following bounds hold: for τ, σ varying on compacts not containing zero:

$$|r_K(s,t)| \le \frac{\pi}{\tau} + \frac{\pi}{\sigma}, \quad |\partial_i r_K(s,t)| \le \frac{\pi}{2\tau} + \frac{\pi}{2\sigma} + \frac{\pi^2}{4\tau^2} + \frac{\pi^2}{4\sigma^2},$$
$$|\partial_{ij} r_K(s,t)| \le c\left(\frac{1}{\tau} + \frac{1}{\sigma} + \frac{1}{\tau^2} + \frac{1}{\sigma^2} + \frac{1}{\tau^3} + \frac{1}{\sigma^3}\right), \tag{5.13}$$

where $i, j = s, t$ and c is some constant; and

$$V_K^2(t) \leq \frac{1}{2}\left[1 + \frac{\pi}{2t}\right]. \tag{5.14}$$

Besides, the same convergences hold for $\overline{T}_K$, and for $\overline{T}'_K$ at least for s, t varying on any compact interval contained in $[t_0, \infty)^2$ (t_0 large enough), see Lemma 5.6. This is enough for our purposes. Therefore, T_K, $\overline{T}_K$ converge to centered Gaussian processes T and $\overline{T}$ on the positive real axis having covariances given by (5.12) and

$$\overline{r}(s,t) = \frac{\mathrm{sc}(\tau) + \mathrm{sc}(\sigma)}{\sqrt{1+\mathrm{sc}(2s)}\sqrt{1+\mathrm{sc}(2t)}}$$

respectively. See Azaïs & León [9] and also Granville & Wigman [37]

Remark 5.5. *On the proof of the CLT we will be concerned with large values of s and t, so σ will be large, thus $\overline{r}(s,t) \approx sc(\tau)$.*

5.3.3 Chaining

In this item we write the processes T_K and T on the same probability space. Furthermore, we establish an isonormal framework.

Assume that $B = (B_\lambda : 0 \leq \lambda \leq 1)$ is a Wiener process (Brownian Motion) in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and define the processes $\tilde{T}_K$ and $\tilde{T}$ by

$$\tilde{T}_K(t) = \int_0^1 \sum_{n=1}^{K} \cos\left(\frac{n}{K}t\right) \mathbb{I}\left\{\left[\frac{n-1}{K}, \frac{n}{K}\right)\right\}(\lambda) dB_\lambda$$

where $\mathbb{I}A$ is the indicator (or characteristic) function of the set A, and

$$\tilde{T}(t) = \int_0^1 \cos(\lambda t)\, dB_\lambda$$

It is easy to verify, using the isometric property of the stochastic integral with respect to B, that these processes have the same distributions than the classical trigonometric polynomial and its limit respectively. Indeed, for instance, for the covariance function of $\tilde{T}_K$ we have

$$cov(\tilde{T}_K(s), \tilde{T}_K(t)) = \mathbb{E} \int_0^1 \sum_{n=1}^{K} \cos\left(\frac{n}{K}s\right) \mathbb{I}\left\{\left[\frac{n-1}{K}, \frac{n}{K}\right)\right\}(\lambda) dB_\lambda$$
$$\cdot \int_0^1 \sum_{m=1}^{K} \cos\left(\frac{m}{K}t\right) \mathbb{I}\left\{\left[\frac{m-1}{K}, \frac{m}{K}\right)\right\}(\lambda') dB_{\lambda'}$$

and by the isometric properties, this equals

$$\int_0^1 \sum_{n=1}^{K} \cos\left(\frac{n}{K}s\right) \mathbb{I}\left\{\left[\frac{n-1}{K}, \frac{n}{K}\right)\right\}(\lambda) \cdot \sum_{m=1}^{K} \cos\left(\frac{m}{K}t\right) \mathbb{I}\left\{\left[\frac{m-1}{K}, \frac{m}{K}\right)\right\}(\lambda) d\lambda$$

$$= \int_0^1 \sum_{n=1}^{K} \cos\left(\frac{n}{K}s\right) \cos\left(\frac{n}{K}t\right) \mathbb{I}\left\{\left[\frac{n-1}{K}, \frac{n}{K}\right)\right\}(\lambda) d\lambda$$

$$= \frac{1}{K} \sum_{n=1}^{K} \cos\left(\frac{n}{K}s\right) \cos\left(\frac{n}{K}t\right) = r_K(s,t).$$

The other cases are analogous.

Hence, the processes $\tilde{T}_K$, $\tilde{T}$ provide a joint representation of T_K and T.

In this way, we have all the processes of interest defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ associated with the Brownian Motion B. In particular, the isonormal process is given by integration with respect to B, namely, the map

$$h \mapsto B(h) := \int_0^1 h dB,$$

defines an isometric between $L^2([0,1])$ and a Gaussian subspace of $L^2(\Omega, \sigma(B), \mathbb{P})$.

We can also compute the covariances, and thus L^2 distances., between T_K and its limit T. Indeed, the cross correlation between $\overline{T}_K$ and $\overline{T}$ is given by

$$\rho_K(s,t) := \mathbb{E}\,\overline{T}_K(s)\overline{T}(t)$$

$$= \frac{1}{V_K(s)V(t)} \mathbb{E} \int_0^1 \sum_{n=1}^{K} \cos\left(\frac{n}{K}s\right) \mathbb{I}\left\{\left[\frac{n-1}{K}, \frac{n}{K}\right)\right\}(\lambda) dB_\lambda \cdot \int_0^1 \cos(\lambda' t)\, dB_{\lambda'}$$

$$= \frac{1}{V_K(s)V(t)} \sum_{n=1}^{K} \int_{\frac{n-1}{K}}^{\frac{n}{K}} \cos\left(\frac{n}{K}s\right) \cos(\lambda t) d\lambda$$

$$= \frac{1}{V_K(s)V(t)} \frac{1}{2} \sum_{n=1}^{K} \int_{\frac{n-1}{K}}^{\frac{n}{K}} \left[\cos\left(\frac{n}{K}s - \lambda t\right) + \cos\left(\frac{n}{K}s + \lambda t\right)\right] d\lambda$$

$$= \frac{1}{\sqrt{1 + c_K(2s)}\sqrt{1 + \mathrm{sc}(2t)}} \sum_{n=1}^{K} \int_0^{1/K} \left[\cos\left(\frac{n}{K}\tau - vt\right) + \cos\left(\frac{n}{K}\sigma - vt\right)\right] dv.$$

where we applied Equation (5.10) and the change of variable $\lambda \mapsto \frac{n}{K} - \lambda$ in the last equality. It follows that, see Azaïs & León [9], $\rho_K(s,t) \longrightarrow \overline{r}(s,t)$ uniformly

on off-diagonal compacts and so happens with the derivatives of $\rho_K(s,t)$ to the respective derivatives of $\overline{r}(s,t)$. Further, these functions are bounded by $c(1/\tau + 1/\sigma)$.

5.3.4 Wiener Chaos decomposition

Following Kratz & León [49] we can establish the expansion of the number of roots of trigonometric polynomials in the Wiener Chaos.

For $\alpha \in [0, 1/2]$, denote $[0, K\pi]_{-\alpha} = [(K\pi)^\alpha, K\pi - (K\pi)^\alpha]$, see Lemma 5.1 below.

Theorem 5.5. *The following expansion holds in* L^2

$$\frac{N_{\overline{T}_K}([0,K\pi]_{-\alpha}) - \mathbb{E}\, N_{\overline{T}_K}([0,K\pi]_{-\alpha})}{\sqrt{K\pi}} = \sum_{q=1}^{\infty} I_q^{\overline{T}_K}([0,K\pi]_{-\alpha})$$

where

$$I_q^{\overline{T}_K}([0,K\pi]_{-\alpha}) = \frac{1}{\sqrt{K\pi}} \int_{[0,K\pi]_{-\alpha}} f_q\left(\overline{T}_K(s), \overline{\overline{T}'_K}(s)\right) v_K(s) ds$$

where $v_K(s) := \sqrt{\overline{r}_K^{(11)}(s,s)} = \sqrt{\partial_{st}\overline{r}_K(s,s)}$ *is the standard deviation of* $\overline{T}'_K(s)$, $\overline{\overline{T}'_K}(s) = \overline{T}'_K(s)/v_K(s)$, *and*

$$f_q(x,y) = \sum_{\ell=0}^{\lfloor q/2 \rfloor} b_{q-2\ell} a_{2\ell} H_{q-2\ell}(x) H_{2\ell}(y),$$

being $a_{2\ell}$ *the Hermite coefficients of the absolute value function* $|\cdot|$ *and* $b_{q-2\ell}$ *are obtained as limits of the Hermite coefficients of the Gaussian density as the variance tends to zero.*

Remark 5.6. *It is well known that*

$$b_k = \frac{1}{k!\sqrt{2\pi}} H_k(0), \quad a_{2\ell} = \sqrt{\frac{2}{\pi}} \frac{(-1)^{\ell+1}}{2^\ell \ell!(2\ell-1)}.$$

Proof. First, since the zeros are isolated, formally, we can write Kac Formula for the number of zeros

$$N_{T_K} = N_{\overline{T}_K} = \int_{[0,K\pi]_{-\alpha}} \delta_0\left(\overline{T}_K(s)\right) \left|\overline{\overline{T}'_K}(s)\right| v_K(s) ds \tag{5.15}$$

Next, in order to give precise meaning to the last expression, we approximate the delta mass at 0 by Gaussian kernels φ_η (centered, with variance η^2), we obtain

$$N^\eta_{\overline{T}_K} := \int_{[0,K\pi]_{-\alpha}} \varphi_\eta(\overline{T}_K(s)) \left|\overline{\overline{T}'_K}(s)\right| v_K(s)ds \tag{5.16}$$

Since $\varphi_\eta \in L^2(\varphi(s)ds)$, $N^\eta_{\overline{T}_K}$ has the following chaotic expansion (see Part 4 of Lemma 5.5)

$$N^\eta_{\overline{T}_K} = \sum_{q=0}^{\infty} \sum_{\ell=0}^{\lfloor q/2 \rfloor} b^\eta_{q-2\ell} a_{2\ell} \int_{[0,K\pi]_{-\alpha}} H_{q-2\ell}\left(\overline{T}_K(s)\right) H_{2\ell}\left(\overline{\overline{T}'_K}(s)\right) v_K(s)ds \tag{5.17}$$

where b^η_k are the Hermite coefficients of φ_η.

Now, the idea is to pass to the limit, with $\eta \to 0$ (K fixed), in this expansion in order to obtain the expansion for $N_{\overline{T}_K}$.

Part 3 of Lemma 5.5 shows that $N^\eta_{\overline{T}_K} \to N_{\overline{T}_K}$ in L^2, so we look at the L^2 limit of the right hand side of Equation (5.17).

First, observe that $b^\eta_k \to_\eta b_k$ (non-random), and that this is the only ingredient depending on η. Besides, the necessary domination is given by Fatou's Lemma

$$\sum_{q=0}^{Q} \mathbb{E}\left[\sum_{\ell=0}^{\lfloor q/2 \rfloor} b_{q-2\ell} a_{2\ell} \int_{[0,K\pi]_{-\alpha}} H_{q-2\ell}(\overline{T}_K(s)) H_{2\ell}(\overline{\overline{T}'_K}(s)) v_K(s)ds\right]^2$$
$$\leq \liminf_{\eta\to 0} \mathbb{E}\,[N^\eta_{\overline{T}_K}]^2 = \mathbb{E}\, N^2_{\overline{T}_K}.$$

Therefore, the right hand side has a limit, say $\mathcal{N}$, in L^2 (with $b_{q-2\ell}$ instead of $b^\eta_{q-2\ell}$). It remains to show that this limit is, effectively, $N_{\overline{T}_K}$.

The result follows writing

$$\|N_{\overline{T}_K} - \mathcal{N}\|^2_{L^2} \leq 2\left[\|N_{\overline{T}_K} - N^\eta_{\overline{T}_K}\|^2_{L^2} + \|N^\eta_{\overline{T}_K} - \mathcal{N}\|^2_{L^2}\right].$$

The first term in the right hand side tends to zero by Part 4 of Lemma 5.5. To show that the second term tends to zero, consider its chaotic expansion

$$N^\eta_{\overline{T}_K} - \mathcal{N} = \sum_{q=0}^{\infty} \sum_{\ell=0}^{\lfloor q/2 \rfloor} \left(b^\eta_{q-2\ell} - b_{q-2\ell}\right) a_{2\ell}\, J_q$$

where we denote $J_q = \int_{[0,K\pi]_{-\alpha}} H_{q-2\ell}(\overline{T}_K(s)) H_{2\ell}(\overline{\overline{T}'_K}(s)) v_K(s)ds$, note that J_q

does not depend on η. Then, for each Q we have

$$\|N^{\eta}_{\overline{T}_K} - \mathcal{N}\|^2_{L^2} \leq 3 \left[\sum_{q=0}^{Q} \mathbb{E} \left[\sum_{\ell=0}^{\lfloor q/2 \rfloor} (b^{\eta}_{q-2\ell} - b_{q-2\ell}) a_{2\ell} J_q \right]^2 + \sum_{q=Q+1}^{\infty} \mathbb{E} \left[\sum_{\ell=0}^{\lfloor q/2 \rfloor} b_{q-2\ell} a_{2\ell} J_q \right]^2 + \sum_{q=Q+1}^{\infty} \mathbb{E} \left[\sum_{\ell=0}^{\lfloor q/2 \rfloor} b^{\eta}_{q-2\ell} a_{2\ell} J_q \right]^2 \right]$$

Now, take limit as $\eta \to 0$, the first term tends to 0 since it is a finite sum and $b^{\eta}_{q-2\ell} \to_{\eta} b_{q-2\ell}$; the second one does not depend on η; the third term is $\|P_Q(N^{\eta}_{\overline{T}_K})\|^2$, where P_Q is the orthogonal projection of the L^2 random variable $N^{\eta}_{\overline{T}_K}$ on the subspace $\oplus^{\infty}_{Q+1}\mathcal{H}_q$, thus it converges with $\eta \to 0$ to $\|P_Q(N_{\overline{T}_K})\|^2$ and tends to 0 when $Q \to \infty$.

This proves the theorem. □

Remark 5.7. *As shown in the Section 5.3.3 , $\overline{T}_K(t)$ and $\overline{\overline{T}'_K}(t)$ can be written as $B(h)$ and $B(h')$ for*

$$h(s, \lambda, K) = \frac{1}{V_K(s)} \sum_{n=1}^{K} \cos\left(\frac{n}{K} s\right) \mathbb{I}\left\{ \left[\frac{n-1}{K}, \frac{n}{K} \right) \right\} (\lambda)$$

and $h' = \partial_s h(s, K)/\|\partial_s h(s, K)\|_2$. Then

$$H_{q-2\ell}(\overline{T}_K(t)) H_{2\ell}(\overline{\overline{T}'_K}(t)) = H_{q-2\ell}(B(h)) H_{2\ell}(B(h')).$$

Since $\overline{T}_K(t)$ and $\overline{\overline{T}'_K}(t)$ are orthogonal, so are by the isometric property h and h', that is, $\int_0^1 h(\lambda)h'(\lambda)d\lambda = 0$. Therefore, the multiplication formula (5.2) and Remark 5.1 show that

$$H_{q-2\ell}(\overline{T}_K(t)) H_{2\ell}(\overline{\overline{T}'_K}(t)) = I^B_q(h^{\otimes q-2\ell} \otimes h'^{\otimes 2\ell}) \in \mathcal{H}_q.$$

Now, using the Stochastic Fubini's Theorem, see Peccati & Taqqu [67, Section 5.13], we have

$$I^{\overline{T}_K}_q([0, K\pi]_{-\alpha}) = I^B_q \left(\frac{1}{\sqrt{K\pi}} \int_{[0,K\pi]_{-\alpha}} \sum_{\ell=0}^{\lfloor q/2 \rfloor} b_{q-2\ell} a_{2\ell} h^{\otimes q-2\ell} \otimes h'^{\otimes 2\ell} v_K(s) ds \right) \in \mathcal{H}_q.$$

Thus $I^{\overline{T}_K}_q([0, K\pi]_{-\alpha}) = I^B_q(g_q)$ for $g_q \in L^2([0,1]^q)$ given by

$$g_q(\boldsymbol{\lambda}, K) = \frac{1}{\sqrt{K\pi}} \int_{[0,K\pi]_{-\alpha}} \sum_{\ell=0}^{\lfloor q/2 \rfloor} b_{q-2\ell} a_{2\ell} (h^{\otimes q-2\ell} \otimes h'^{\otimes 2\ell})(\boldsymbol{\lambda}) v_K(s) ds, \tag{5.18}$$

with $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_q)$.

In conclusion, the random variable $I^{\overline{T}_K}_q$ belongs to the q-th Wiener Chaos $\mathcal{H}_q$. In particular, they are orthogonal for different values of q.

5.3.5 Asymptotic variance

The main idea is to take advantage of the fact that the covariances of $\overline{T}_K$ and Y_K are very similar one to each other for large values of s, t, even $\overline{T}_K$ (and T_K) being non-stationary processes.

In fact, remember that the covariance of Y_K is $c_K(\tau)$ and that of $\overline{T}_K$ is $(c_K(\tau)+c_K(\sigma))/\sqrt{1+c_K(2s)}\sqrt{1+c_K(2t)}$.

This idea is supported by the following lemma, where $[0, K\pi]^c_{-\alpha} = [0, K\pi] \setminus [0, K\pi]_{-\alpha}$.

Lemma 5.1. *For $0 < \alpha < 1/2$, we have*

$$\frac{N_{\overline{T}_K}\left([0, K\pi]^c_{-\alpha}\right) - \mathbb{E}\, N_{\overline{T}_K}\left([0, K\pi]^c_{-\alpha}\right)}{\sqrt{K\pi}} \longrightarrow 0$$

in probability, as $K \to \infty$. The same result holds true also for $\overline{T}$.

Proof. Let us look at the interval $[0, (K\pi)^\alpha]$, the other one is analogous. First, we use Markov inequality to bound the probability by an expression involving the expectation of the number of roots, that is, for given $\varepsilon > 0$ we have

$$\mathbb{P}\left(\left|\frac{N_{\overline{T}_K} - \mathbb{E}\, N_{\overline{T}_K}}{\sqrt{K\pi}}\right| > \varepsilon\right) \leq \frac{\mathbb{E}\,|N_{\overline{T}_K} - \mathbb{E}\, N_{\overline{T}_K}|}{\varepsilon\sqrt{K\pi}} \leq \frac{2\mathbb{E}\, N_{\overline{T}_K}}{\varepsilon\sqrt{K\pi}}$$

thus, it is enough to show that $\mathbb{E}\, N_{\overline{T}_K}([0, (K\pi)^\alpha])/\sqrt{K\pi} \to 0$.

With this aim, we use the first order Rice formula.

$$\begin{aligned}\mathbb{E}\, N_{\overline{T}_K}([0, (K\pi)^\alpha]) &= \int_0^{(K\pi)^\alpha} \mathbb{E}\left[\overline{T}'_K(t) \mid \overline{T}_K(t) = 0\right] p_{\overline{T}_K(t)}(0)dt \\ &= \frac{1}{\pi}\int_0^{(K\pi)^\alpha} \left[c''_K(2t) - c''_K(0) - \frac{(c'_K(2t) - c'_K(0))^2}{1 + c_K(2t)}\right]^{1/2} \frac{dsdt}{\sqrt{1+c_K(2t)}}\end{aligned}$$

It is easy to see that the integrand is bounded: the covariances and its derivatives are bounded by Equation (5.14), and the denominator is bounded away from zero by Lemma 5.6. Hence, the result follows. The proof for the limit process $\overline{T}$ is exactly the same, replacing c_K by sc. □

This Lemma shows that in order to obtain the desired CLT, it suffices to prove that

$$\frac{N_{\overline{T}_K}\left([0, K\pi]_{-\alpha}\right) - \mathbb{E}\, N_{\overline{T}_K}\left([0, K\pi]_{-\alpha}\right)}{\sqrt{K\pi}} \Rightarrow N(0, V^2).$$

Following the arguments in Azaïs & León [9] we obtain the asymptotic variance of the number of zeros of $\overline{T}_K$ on $[0, K\pi]_{-\alpha}$ as $K \to \infty$. Further, in Lemma 5.4 we show that the limit variance of $N_{\overline{T}_K}([0, K\pi]_{-\alpha})$ coincide with that of $N_{\overline{T}}([0, K\pi]_{-\alpha})$, and that both coincide with the limit variance of the number of zeros of the stationary Gaussian process X, $N_X([0, K\pi]_{-\alpha})$.

The two following lemmas give an uniform upper bound (on K) for these variances. The first one deals with the difficult part: when integrating close to the diagonal.

Lemma 5.2. *For fixed $a > 0$, the variances of $N_{\overline{T}_K}([t, t+a])/\sqrt{K\pi}$ remain bounded uniformly on K for all $t \in [t_0, K\pi - t_0]$ for t_0 large enough. Besides, the same result holds true also for $\overline{T}$.*

Proof. Let t_0 be as in Lemma 5.6 to ensure that the standard deviation v, v_K of $\overline{T}'(t)$ and $\overline{T}'_K(t)$ are bounded away from zero.

In the first place, note that, it suffices to bound the second factorial moment of the number of zero up-crossings $U = U([t, t+a])$ of $\overline{T}_K$ (resp. $\overline{T}$) on the interval $[t, t+a]$. Indeed, by the continuity of the paths it is easy to see that $N \leq 2U + 1$, besides, similar computations as in the proof of Lemma 5.1 show that $\mathbb{E}\, U/\sqrt{K\pi}$ tends to zero as $K \to \infty$.

Then, we consider the second order Rice Formula, see Proposition 4.1 in Azaïs & Wschebor.

$$\begin{aligned}\mathbb{E}\left(U_{[2]}\right) &= \mathbb{E}\left(U(U-1)\right) \\ &= \int_t^{t+a}\int_t^{t+a} \mathbb{E}\left[\overline{T}'^{+}_K(s)\overline{T}'^{+}_K(t) \mid \overline{T}_K(s) = \overline{T}_K(t) = 0\right] \cdot p_{\overline{T}_K(s),\overline{T}_K(s)}(0,0)dsdt \\ &= \int_t^{t+a}\int_t^{t+a} \mathbb{E}\left[\Theta^+(s)\Theta^+(t)\right] \cdot p_{\overline{T}_K(s),\overline{T}_K(s)}(0,0)dsdt\end{aligned}$$

where the Θ's are the (Gaussian) regression variables, that is

$$\Theta(s) = \overline{T}'_K(s) - \alpha(s)\overline{T}_K(s) - \beta(s)\overline{T}_K(t)$$

where $\alpha(s), \beta(s)$ are chosen such that $\Theta(s)$ is independent from $\overline{T}_K(s)$ and $\overline{T}_K(t)$. Thus, see Appendix 6.5.5,

$$\alpha(s) = -\overline{r}_K(s,t)\beta(s), \quad \beta(s) = \frac{\overline{r}_K^{(10)}(s,t)}{1 - \overline{r}_K^2(s,t)}.$$

Similar formulas hold true for $\Theta(t)$ and for $\overline{T}$.

Since we are looking at the roots of the polynomials, we can use the arguments in Azaïs & Wschebor [12, Eq. 4.14]. Indeed, the conditional expectations of

$\overline{T}_K(s)$ and $\overline{T}_K(t)$ vanish, thus we can bound the expectation in the last equation by the sum of the conditional variances, that is by the sum of the variances of $\Theta(s)$ and of $\Theta(t)$. Indeed

$$\begin{aligned}\mathbb{E}\left[\Theta^+(s)\Theta^+(t)\right] &\leq \frac{1}{4}\mathbb{E}\left[\Theta(s)+\Theta(t)\right]^2 \\ &\leq \frac{1}{4}\mathbb{E}\left[\Theta(s)+\Theta(t)-\mathbb{E}\,\Theta(s)-\mathbb{E}\,\Theta(t)\right]^2 \\ &\leq \frac{1}{2}\left[var(\Theta(s))+var(\Theta(t))\right] = var(\Theta(s)).\end{aligned}$$

Hence, the conditional variance of $\overline{T}_K(s)$ is

$$var(\Theta(s)) = \overline{r}_K^{(11)}(s,s) - \frac{(\overline{r}_K^{(10)})^2(s,t)}{1-\overline{r}_K^2(s,t)}, \tag{5.19}$$

which follows directly from its definition. Note that this expression is similar to the one in the stationary case. So the integrand is

$$\frac{\overline{r}_K^{(11)}(s,s)(1-\overline{r}_K^2(s,t)) - (\overline{r}_K^{(10)})^2(s,t)}{1-\overline{r}_K^2(s,t)} \cdot \frac{1}{\pi\sqrt{1-\overline{r}_K^2(s,t)}},$$

and changing variables from (s,t) to (τ,σ), the domain of integration is $[0,a]\times[2t,2t+2a]$.

Both, numerator and denominator tend to zero as $\tau\to 0$ (diagonal points). Let us look them carefully.

Let us consider the denominator of Equation (5.19) (the first factor in the r.h.s. of the latter equation) first, after simplifying the denominators (of the $\overline{r}$'s), we have that the denominator is

$$\begin{aligned}&(1+c_K(2s))(1+c_K(2t)) - (c_K(\tau)+c_K(\sigma))^2 \\ &\qquad = (1+c_K(\sigma-\tau))(1+c_K(\sigma+\tau)) - (c_K(\tau)+c_K(\sigma))^2 \\ &= [1-c_K^2(\tau)] + [c_K(2s)c_K(2t) - c_K^2(\sigma)] + [c_K(2s)+c_K(2t)-2c_K(\tau)c_K(\sigma)] \\ &\qquad := S_1+S_2+S_3,\end{aligned}$$

with self evident notation. As above, the same formula hold for $\overline{T}$ with c_K replaced by sc.

We take Taylor expansions at 0 and σ with increment τ.

$$\begin{aligned}S_1 &= 1-\left(1+c_K'(0)\tau+c_K''(0)\frac{\tau^2}{2}+\theta_{K0}(\tau)\right)^2 \\ &= -c_K''(0)\tau^2 - 2\theta_{K0}(\tau) - \frac{c_K''^2(0)}{4}\tau^4 - \theta_{K0}(\tau)(\theta_{K0}(\tau)+c_K''(0)\tau^2)\end{aligned}$$

where we used the fact that $c'_K(0) = 0$.

$$\begin{aligned}
S_2 &= c_K(\sigma-\tau)c_K(\sigma+\tau) - c_K^2(\sigma) \\
&= \left[c_K(\sigma) - c'_K(\sigma)\tau + c''_K(\sigma)\frac{\tau^2}{2} + \theta_{K\sigma}(-\tau)\right] \\
&\quad \cdot \left[c_K(\sigma) + c'_K(\sigma)\tau + c''_K(\sigma)\frac{\tau^2}{2} + \theta_{K\sigma}(\tau)\right] - c_K^2(\sigma) \\
&= \tau^2\left[c_K(\sigma)c''_K(\sigma) - c'^2_K(\sigma)\right] + c_K(\sigma)\big(\theta_{K\sigma}(\tau) + \theta_{K\sigma}(-\tau)\big) \\
&\quad + \frac{c''^2_K(\sigma)}{4}\tau^4 + c'_K(\sigma)\tau\big(\theta_{K\sigma}(\tau) + \theta_{K\sigma}(-\tau)\big) \\
&\quad + c''_K(\sigma)\frac{\tau^2}{2}\big(\theta_{K\sigma}(\tau) + \theta_{K\sigma}(-\tau)\big) + \theta_{K\sigma}(-\tau)\theta_{K\sigma}(\tau)
\end{aligned}$$

(the θ's are of order three). Finally

$$\begin{aligned}
S_3 &= c_K(\sigma-\tau) - c_K(\tau)c_K(\sigma) + c_K(\sigma+\tau) - c_K(\tau)c_K(\sigma) \\
&= \tau^2\left[c''_K(\sigma) - c_K(\sigma)c''_K(0)\right] + \theta_{K\sigma}(-\tau) + \theta_{K\sigma}(\tau) + c_K(\sigma)\theta_{K0}(\tau).
\end{aligned}$$

Therefore, the denominator is

$$\begin{aligned}
S_1 + S_2 + S_3 &= \tau^2\left[(c''_K(\sigma) - c''_K(0))(1 + c_K(\sigma)) - c'^2_K(\sigma)\right] \\
&\quad + (c_K(\sigma) - 2)\theta_{K0}(\tau) + (1 + c_K(\sigma))\big(\theta_{K\sigma}(\tau) + \theta_{K\sigma}(-\tau)\big) \\
&\quad + \frac{\tau^4}{4}\left[c''^2_K(\sigma) - c''^2_K(0)\right] + o(\tau^4). \quad (5.20)
\end{aligned}$$

Let us look now at the numerator of Equation (5.19). We begin with the term $(\overline{r}_K^{(10)})^2(s,t) = \mathbb{E}\,\overline{T}'_K(s)\overline{T}_K(t)$. After simplifying the denominator, we have

$$\begin{aligned}
\overline{r}_K^{(10)}(s,t) &= c'_K(\sigma) - c'_K(\tau) - \frac{c'_K(2s)}{1 + c_K(2s)}(c_K(\tau) + c_K(\sigma)) \\
&= \frac{1}{1 + c_K(2s)}\Big[(c'_K(\sigma) - c'_K(\tau))(1 + c_K(\sigma-\tau)) - c'_K(\sigma-\tau)(c_K(\tau) + c_K(\sigma))\Big]
\end{aligned}$$

We look at the factor in brackets, again by a Taylor expansion, for the first term, recalling that $c'_K(0) = 0$, we have

$$\begin{aligned}
&(c'_K(\sigma) - c''_K(0)\tau - \theta'_{K0}(\tau))\left[1 + c_K(\sigma) - c'_K(\sigma)\tau + c''_K(\sigma)\frac{\tau^2}{2} + \theta_{K\sigma}(-\tau)\right] \\
&\quad = c'_K(\sigma)(1 + c_K(\sigma)) + \tau\Big(-c''_K(0)(1 + c_K(\sigma)) - c'^2_K(\sigma)\Big) \\
&\quad + \tau^2\left[\frac{c'_K(\sigma)c''_K(\sigma)}{2} + c''_K(0)c'_K(\sigma)\right] - \theta'_{K0}(\tau)(1 + c_K(\sigma)) \\
&\quad - c''_K(0)c''_K(\sigma)\frac{\tau^3}{2} + c'_K(\sigma)\theta_{K\sigma}(-\tau) + o(\tau^3)
\end{aligned}$$

For the second term

$$\Big(c'_K(\sigma) - c''_K(\sigma)\tau + \theta'_{K\sigma}(-\tau)\Big)\Big(1 + c'_K(0)\tau + c''_K(0)\frac{\tau^2}{2} + \theta_{K0}(\tau) + c_K(\sigma)\Big)$$
$$= c'_K(\sigma)(1 + c_K(\sigma)) + \tau\Big(- c''_K(\sigma)(1 + c_K(\sigma))\Big)$$
$$+ \tau^2\frac{c'_K(\sigma)c''_K(0)}{2} + \theta'_{K\sigma}(-\tau)(1 + c_K(\sigma))$$
$$+ c'_K(\sigma)\theta_{K0}(\tau) - c''_K(\sigma)c''_K(0)\frac{\tau^3}{2} + o(\tau^3)$$

Subtracting the two previous formulas, the factor into brackets is

$$\tau\Big((c''_K(\sigma) - c''_K(0))(1 + c_K(\sigma)) - c'^2_K(\sigma)\Big) + \frac{\tau^2}{2}\left[c'_K(\sigma)c''_K(\sigma) + c'_K(\sigma)c''_K(0)\right]$$
$$- (\theta'_{K0}(\tau) + \theta'_{K\sigma}(-\tau)(1 + c_K(\sigma)))$$
$$+ c'_K(\sigma)\left(\theta_{K\sigma}(-\tau) - \theta_{K0}(\tau)\right) + o(\tau^3) \quad (5.21)$$

(the θs are of order three). Compare with (5.20).

It remains only to see $\overline{r}^{(11)}_K(s,t)$. From the Auxiliary Computations, see Equation (5.24), we have

$$\overline{r}^{(11)}_K(s,t) = \frac{1}{1 + c_K(2s)}\left[c''_K(\sigma) - c''(\tau) - \frac{c'^2_K(2s)}{1 + c_K(2s)}\right]$$
$$= \frac{1}{(1 + c_K(2s))^2}\left[(c''_K(\sigma) - c''(\tau))(1 + c_K(\sigma - \tau)) - c'^2_K(\sigma - \tau)\right]$$

The first term of the factor in brackets is approximated by Taylor expansion by

$$\Big(c''_K(\sigma) - c''_K(0) - \theta''_{K0}(\tau)\Big)\Big(1 + c_K(\sigma) - c'_K(\sigma)\tau + c''_K(\sigma)\frac{\tau^2}{2} + \theta_{K\sigma}(-\tau)\Big)$$
$$= (c''_K(\sigma) - c''_K(0))(1 + c_K(\sigma)) - c'_K(\sigma)(c''_K(\sigma) - c''_K(0))\tau - \theta''_{K0}(\tau)(1 + c_K(\sigma))$$
$$+ (c''_K(\sigma) - c''_K(0))c''_K(\sigma)\frac{\tau^2}{2} + c'_K(\sigma)\theta''_{K0}(\tau)\tau$$
$$+ (c''_K(\sigma) - c''_K(0))\theta_{K\sigma}(-\tau) - \theta''_{K0}(\tau)c''_K(\sigma)\frac{\tau^2}{2}$$
$$- \theta''_{K0}(\tau)\theta_{K\sigma}(-\tau)$$

The second term is

$$\Big(c'_K(\sigma) - c''_K(\sigma)\tau + \theta'_{K\sigma}(-\tau)\Big)^2$$
$$= c'^2_K(\sigma) - 2c'_K(\sigma)c''_K(\sigma)\tau + c''^2_K(\sigma)\tau^2 + 2c'_K(\sigma)\theta'_{K\sigma}(-\tau)$$
$$- 2c''_K(\sigma)\theta'_{K\sigma}(-\tau)\tau + \theta'^2_{K\sigma}(-\tau)$$

Therefore

$$\overline{r}_K^{(11)}(s,s) = \frac{1}{1+c_K^2(2s)}\left[(c_K''(\sigma)-c_K''(0))(1+c_K(\sigma)) - c_K'^2(\sigma) \right. \\ \left. + c_K'(\sigma)(c_K''(\sigma)+c_K''(0))\tau - \theta_{K0}''(\tau)(1+c_K(\sigma)) + o(\tau)\right] \quad (5.22)$$

Finally, putting together Equations (5.21) and (5.22), the numerator is

$$\frac{1}{(1+c_K^2(2s))^2}\left[\Big((c_K''(\sigma)-c_K''(0))(1+c_K(\sigma)) - c_K'^2(\sigma) \right. \\ + c_K'(\sigma)(c_K''(\sigma)+c_K''(0))\tau - \theta_{K0}''(\tau)(1+c_K(\sigma)) + o(\tau)\Big)\tau^2 \\ \cdot \Big((c_K''(\sigma)-c_K''(0))(1+c_K(\sigma)) - c_K'^2(\sigma)\Big) \\ \left. -\tau^2\Big((c_K''(\sigma)-c_K''(0))(1+c_K(\sigma)) - c_K'^2(\sigma)\Big)^2 + o(\tau^3)\right]$$

Simplifying

$$\frac{1}{(1+c_K^2(2s))^2}\left[\Big(c_K'(\sigma)(c_K''(\sigma)+c_K''(0))\tau - \theta_{K0}''(\tau)(1+c_K(\sigma)) + o(\tau)\Big)\tau^2 \right. \\ \left. \Big((c_K''(\sigma)-c_K''(0))(1+c_K(\sigma)) - c_K'^2(\sigma)\Big) + o(\tau^3)\right]$$

This shows that the leading term in the numerator of Equation (5.19) is of order three, the same as in the denominator, thus, the integral ($\mathbb{E}\, U_{[2]}$) is convergent. Again, the same holds for $\overline{T}$, note that the computations involve c_K but not its specific form.

The constants in the higher order terms depend on $c_K(0)$, $c_K(\sigma)$, $\theta_{K0}(\tau)$, $\theta_{K\sigma}(\pm\tau)$ and their first and second order derivatives. Equation (5.13) shows that $c_K(\sigma)$ and its derivatives are bounded uniformly on K; besides a direct computation from Equation (5.8) shows that $c_K(0)=1$, $c_K'(0)=0$ and $c_K''(0) = -\frac{(K+1)(2K+1)}{6K^2}$, therefore they are uniformly bounded on K.

Finally, the terms θ_{K0} and $\theta_{K\sigma}$ are also bounded uniformly on K. Indeed, $\theta_{K\sigma}(\tau)$ is the remainder of second order Taylor's formula, therefore it can be written in the form (Lagrange's form) $\theta_{k\sigma}(\tau) = c_K''(\xi)\tau^3/6$ with $\xi \in (\sigma, \sigma+\tau)$, in our case

$$\theta_{K\sigma}(\tau) = \frac{1}{N}\left(-\sum_{n=1}^{K}\left[\frac{n}{N}\right]^2 \cos(n\xi/N)\right)\frac{\tau^3}{6}$$

Thus, $|\theta_{K\sigma}(\tau)| \leq \tau^3/6$. The remaining cases are analogous. □

Lemma 5.3. *There exists $t_0 > 0$ such that the variances of the normalized number of roots of $\overline{T}_K$ on the interval $[t_0, K\pi - t_0]$*

$$var\left(\frac{N_{\overline{T}_K} - \mathbb{E}\, N_{\overline{T}_K}}{\sqrt{K\pi}}\right)$$

are uniformly bounded on K.

Proof. We follow the proof of the similar assertion in Azaïs & León [9]. Let us denote $J = [t_0, K\pi - (K\pi)^\alpha]$, besides, C stands for an absolute constant whose value may change from one line to another.

Consider Arcones coefficient ψ for $(\overline{T}_K(s), \overline{\overline{T}'}_K(s))$ and $(\overline{T}_K(t), \overline{\overline{T}'}_K(t))$. Recall that ψ involve the covariances of these two vectors which are functions of τ and σ, see the Preliminaries above 74, so $\psi = \psi(\tau, \sigma)$. The computations in the Auxiliary Results, page 97, and Inequalities (5.13) show that, for a given $0 < \rho < 1$, one can choose a and t_0 in order that for $\tau > a$, $\sigma > a$ (thus $t_0 > a/2$) we have $\psi(\tau, \sigma) \leq (const)(1/\tau + 1/\sigma) < \rho < 1$.

Now, write J as the disjoint union of intervals J_ℓ of length greater than a. Therefore $var(N^{\overline{T}_K}(J)) = \sum_{\ell,\ell'} cov(N^{\overline{T}_K}(J_\ell), N^{\overline{T}_K}(J_{\ell'}))$. When $\ell = \ell'$ or $|\ell - \ell'| = 1$ we apply Cauchy-Schwarz Inequality and Lemma 5.2 to show that each term is uniformly bounded on K. Therefore the sum of these terms is $O(K)$, and divided by $K\pi$, it remains bounded uniformly on K.

For the remaining terms, consider the chaotic expansion

$$\frac{N_{\overline{T}_K}(J_\ell) - \mathbb{E}\, N_{\overline{T}_K}(J_\ell)}{\sqrt{K\pi}} = \sum_{q=1}^{\infty} I_q^{\overline{T}_K}(J_\ell),$$

being $I_q^{\overline{T}_K}$ as in Theorem 5.5 replacing $[0, K\pi]_{-\alpha}$ by J_ℓ. Note that the denominator $K\pi$ is included in $I_q^{\overline{T}_K}$.

Therefore, for $q \geq 2$

$$\begin{aligned}
&\sum_{\ell,\ell':|\ell-\ell'|>1} cov\left(N_{\overline{T}_K}(J_\ell), N_{\overline{T}_K}(J_{\ell'})\right) \\
&= \frac{1}{K\pi} \sum_{q=1}^{\infty} \iint_{\cup J_\ell \times J_{\ell'}:|\ell-\ell'|>1} f_q(\overline{T}_K(s), \overline{\overline{T}'}_K(s)) f_q(\overline{T}_K(t), \overline{\overline{T}'}_K(t)) v_K(s) v_K(dt) ds dt \\
&\leq \frac{C}{K\pi} \sum_{q=1}^{\infty} \iint_{|s-t|\geq a} q^2 \rho^{q-2} \left[\frac{1}{\tau} + \frac{1}{\sigma}\right]^2 ds dt \\
&= \frac{C}{K\pi} \iint_{|s-t|\geq a} \left[\frac{1}{\tau} + \frac{1}{\sigma}\right]^2 ds dt \\
&\leq \frac{C}{K\pi} \int_a^{K\pi} \int_{2a}^{2K\pi} \left[\frac{1}{\tau^2} + \frac{1}{\sigma^2}\right] d\tau d\sigma \leq \frac{C}{aK\pi}
\end{aligned}$$

where we use Arcones Inequality and the fact that the coefficients $b_{q-2\ell}, a_{2\ell} \leq 1$ for the first inequality, see Remark 5.6, and we enlarge the domain of integration and use the inequality $(a+b)^2 \leq 2(a^2 + b^2)$ for the second one.

We have to consider separately the case $q = 1$. but note that $b_1 = \varphi(0)H_1(0) = 0$ since $H_1(x) = x$, for all x. Thus $I_1^{\overline{T}_K} = 0$ for all K.

The result follows. □

We are ready to compare the limit variances of $N_{\overline{T}_K}([0, K\pi]_{-\alpha})/\sqrt{K\pi}$, $N_{\overline{T}}([0, K\pi]_{-\alpha})/\sqrt{K\pi}$. and $N_X([0, K\pi]_{-\alpha})/\sqrt{K\pi}$.

Lemma 5.4 (Asymptotic variance)**.** *The variances of* $I_q^{\overline{T}_K}([0, K\pi]_{-\alpha})$, $I_q^{\overline{T}}([0, K\pi]_{-\alpha})$ *and* $I_q^X([0, K\pi]_{-\alpha})$ *have the same limit, denoted by* $\sigma_q^2(0)$*, in particular,* $\sigma_1^2(0) = 0$*. Consequently,*

$$var\left(\frac{N_{\overline{T}_K}([0, K\pi]_{-\alpha}) - \mathbb{E}\, N_{\overline{T}_K}([0, K\pi]_{-\alpha})}{\sqrt{K\pi}}\right) \to_K \sum_{q=2}^{\infty} \sigma_q^2(0),$$

Furthermore,

$$I_q^{\overline{T}_K}([0, K\pi]_{-\alpha}) - I_q^{\overline{T}}([[0, K\pi]_{-\alpha}) \to 0$$

in L^2*.*

Proof. The variance of $I_{qK} := I_q^{\overline{T}_K}([0, K\pi]_{-\alpha})$ is

$$\begin{aligned} var(I_{qK}) &= \mathbb{E}\, I_{qK}^2 \\ &= \frac{1}{K\pi}\int_{(K\pi)^\alpha}^{K\pi-(K\pi)^\alpha}\int_{(K\pi)^\alpha}^{K\pi-(K\pi)^\alpha} \mathbb{E}\left[f_q(\overline{T}_K(s), \overline{T}'_K(s))f_q(\overline{T}_K(t), \overline{T}'_K(t))\right] \\ &\qquad \cdot v_K(s)v_K(t)dsdt \\ &= \frac{1}{K\pi}\int_0^{K\pi-(K\pi)^\alpha}\int_{2(K\pi)^\alpha+\tau}^{2K\pi-2(K\pi)^\alpha+\tau} E(\tau,\sigma)v_K((\sigma-\tau)/2)v_K((\sigma+\tau)/2)d\sigma d\tau \end{aligned}$$

where $E(\tau, \sigma)$ is the expectation written in terms of (τ, σ).

By Equation (5.24), $v_K(u) \to \sqrt{-\mathrm{sc}''(0)} = 1/\sqrt{3}$ as $u \to \infty$. By Mehler Formula, see the Preliminaries in page 74, $E(\tau, \sigma)$ is a polynomial on the covariances of $(\overline{T}_K(s), \overline{T}'_K(s))$ and $(\overline{T}_K(t), \overline{T}'_K(t))$.

Now, the inner integral (w.r.t. σ) divided by $K\pi$ is, applying the mean value theorem, equivalent to the integrand in an intermediate value $\tilde{\sigma}$. Thus, as $K \to \infty$, $\tilde{\sigma} \to \infty$, $c_K(\tilde{\sigma}) \to 0$ and $E(\tau, \sigma)$ converges to the same polynomial on the covariances of the stationary process X and its derivative X' evaluated at s and t (which depend on $\tau = t - s$). Indeed, recall that the covariance of $\overline{T}_K$ is $(c_K(\tau)+c_K(\sigma))/\sqrt{(1 + c_K(2s)(1 + c_K(2t)))}$, thus, this tends to $\mathrm{sc}(\tau)$ as $K \to \infty$.

Using the domination of Lemma 5.3, the result follows.

The second assertion:

$$\|I_{qK} - I_q\|_{L^2}^2 = \mathbb{E}\,(I_{qK} - I_q)^2 = \mathbb{E}\,I_{qK}^2 + \mathbb{E}\,I_q^2 - 2\mathbb{E}\,I_{qK} I_q$$

The first part of this lemma says that the first two terms converge to the same value. The last term, can be computed similarly, and since the cross correlation ρ_K also converges to $\overline{r}$, the limit coincides with the previous one. Thus $\lim_K \|I_{qK} - I_q\|_{L^2}^2 = 0$. □

Remark 5.8. *In spite of the asymptotic equality of their variances, the random variables $I_q^{\overline{T}}([0, K\pi]_{-\alpha})$ and $I_q^X([0, K\pi]_{-\alpha})$ do not approximate each other in L^2 as K grows to infinity. Indeed, let us compute their cross correlation. Thus, consider a second Brownian Motion $W = (W_\lambda : \lambda \in [0, 1])$ independent from B (defined on the same probability space). We can represent X, see Azaïs & León [9, Eq. 9], by*

$$X(t) = \int_0^1 \cos(\lambda t) dB_\lambda + \int_0^1 \sin(\lambda t) dW_\lambda.$$

The cross correlation ρ of $\overline{T}$ and X is

$$\begin{aligned}\rho(s,t) := \mathbb{E}\,\overline{T}(s)X(t) &= \frac{1}{V(s)}\mathbb{E}\int_0^1 \cos(\lambda s) dB_\lambda \cdot \int_0^1 \cos(\lambda' t)\, dB_{\lambda'} \\ &= \frac{1}{V(s)}\int_0^1 \cos(\lambda s)\cos(\lambda t) d\lambda = \frac{1}{V(s)}\frac{1}{2}\int_0^1 [\cos(\lambda\tau) + \cos(\lambda\sigma)]\, d\lambda \\ &= \frac{1}{\sqrt{2}\sqrt{1 + sc(2s)}}(sc(\tau) + sc(\sigma)) \to_K \frac{sc(\tau)}{\sqrt{2}}\end{aligned}$$

where we used the independence of B and W in the first equality and the fact that in the truncated interval, $\sigma \to \infty$.

Therefore, the covariance of $I_q^{\overline{T}}([0, K\pi]_{-\alpha})$ and $I_q^X([0, K\pi]_{-\alpha})$ do not cancel the sum of their variances when computing their L^2 distance analogously to the proof of the second assertion in the last theorem.

5.3.6 CLT

In this section we prove Theorem 1, the Central Limit Theorem for the number of zeros of Classical Trigonometric Polynomials.

For $q = 1$, we saw in Lemma 5.3 that the random variables $I_1^{\overline{T}_K}$ vanish.

Asymptotic Gaussianity of $I_q^{\overline{T}_K}$ for $q > 1$

Now, Remark 5.8 implies that we can not use the L^2 proximity with the stationary process X as in Azaïs & León [9]. Instead, we proceed by the contractions argument, see Preliminaries, page 75, and Peccati & Tudor [68] or Peccati & Taqqu [67].

By Equation (5.18) we know that $I_q^{\overline{T}_K} = I_q^B(g_q)$ with

$$g_q(\boldsymbol{\lambda}, K) = \frac{1}{\sqrt{K\pi}} \int_{[0,K\pi]_{-\alpha}} \sum_{\ell=0}^{\lfloor q/2 \rfloor} b_{q-2\ell} a_{2\ell} (h(s)^{\otimes q-2\ell} \otimes h'(s)^{\otimes 2\ell})(\boldsymbol{\lambda}) v_K(s) ds,$$

and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_q)$.

In order to compute the contractions $g_q \otimes_n g_q$, $n = 1, \dots, q-1$, and the norms $\|g_q(K)\|_2^2$ and $\|g_q \otimes_n g_q(K)\|_2^2$ we have to perform the product of the integral defining g_q with itself two/four times and to integrate that with respect to the λ's.

As by the isometric property of stochastic integral

$$\int_0^1 h(s;\lambda) h(t,\lambda) d\lambda = \overline{r}_K(s,t)$$

we have $h(s)^{\otimes p} \otimes_n h(t)^{\otimes p} = \overline{r}_K(s,t)^n \cdot h(s)^{\otimes p-n} \otimes h(t)^{\otimes p-n}$. Indeed

$$\begin{aligned}
&h(s)^{\otimes p} \otimes_n h(t)^{\otimes p}(x_1, \dots, x_{p-n}, y_1, \dots, y_{p-n}) \\
&\quad = \int_{[0,1]^n} \left[\prod_{k=1}^n h(s;z_k) \prod_{k=1}^{p-n} h(s;x_k) \cdot \prod_{k=1}^n h(t;z_k) \prod_{k=1}^{p-n} h(t;y_k) \right] dz_1 \cdots dz_n \\
&\qquad = \prod_{k=1}^n \left[\int_0^1 h(s;z_k) h(t;z_k) dz_k \right] \prod_{k=1}^{p-n} h(x_k) \prod_{k=1}^{p-n} h(y_k) \\
&\qquad = \overline{r}_K(s,t)^n \cdot h(s)^{\otimes p-n}(x_1, \dots, x_{p-n}) \otimes h(t)^{\otimes p-n}(y_1, \dots, y_{p-n}).
\end{aligned}$$

Similar formulas hold for the terms which include h'. More precisely, by the isometric property, they give factors that are powers of $cov(\overline{T}_K(s), \overline{T'}_K(t)) =: \widetilde{r}'_K(s,t)$ or $cov(\overline{T'}_K(s), \overline{T'}_K(t)) =: \widetilde{r}''_K(s,t)$. These covariances are computed in the Auxiliary Results, see page 97.

Therefore

$$\begin{aligned}
\|g_q(K)\|_2^2 = \frac{1}{K\pi} \int_{[0,K\pi]_{-\alpha}} \int_{[0,K\pi]_{-\alpha}} v_K(s) v_K(t) \sum_{\ell=0}^{q/2} \sum_{\ell'=0}^{q/2} b_{q-2\ell} b_{q-2\ell'} a_{2\ell} a_{2\ell'} \\
\cdot \overline{r}_K(s,t)^{q-2\ell\vee\ell'} \widetilde{r}''_K(s,t)^{2\ell\wedge\ell'} \widetilde{r}'_K(s,t)^{2\ell\vee\ell'-2\ell\wedge\ell'} ds dt
\end{aligned}$$

It is quite tedious to write down the contractions and their norms, but the resulting integrals are quite similar. Let us do it for the case $n = 1$ and q odd (so that $q - 2\ell > 0$), we have

$$\|g_q \otimes_1 g_q(K)\|_2^2 \leq \frac{1}{(K\pi)^2} \int_{[0,K\pi]_{-\alpha}} \int_{[0,K\pi]_{-\alpha}} \int_{[0,K\pi]_{-\alpha}} \int_{[0,K\pi]_{-\alpha}} dsdtds'dt'$$
$$v_K(s)v_K(t)v_K(s')v_K(t') \sum_{\ell,\ell'=0}^{q/2} \sum_{k,k'=0}^{q/2} |b_{q-2\ell}b_{q-2\ell'}a_{2\ell}a_{2\ell'}||b_{q-2k}b_{q-2k'}a_{2k}a_{2k'}|$$
$$\cdot \left|\overline{r}_K(s,t)\overline{r}_K(s',t')\overline{r}_K(s,s')^{q-2\ell\vee\ell'-1}\widetilde{r}''_K(s,s')^{2\ell\wedge\ell'}\widetilde{r}'_K(s,s')^{2\ell\vee\ell'-2\ell\wedge\ell'}\right|$$
$$\cdot \left|\overline{r}_K(t,t')^{q-2k\vee k'-1}\widetilde{r}''_K(t,t')^{2k\wedge k'}\widetilde{r}'_K(t,t')^{2k\vee k'-2k\wedge k'}\right|$$

The inequality is due to the absolute values in the covariances. Observe that in the general case, the exponent of $\overline{r}_K(s,t)$ and $\overline{r}_K(s',t')$ is n, but the sum of the exponents in the factors involving (s,t), (s,s'), and in those involving (s',t'), (t,t'), is q (the total sum of the exponents is $2q$).

In the case of $\|g_q\|_2^2$, since I_q^B is an isometry, it follows that $\|g_q\|_2^2 = var(I_q^{\overline{T}_K}) \to_{K\to\infty} \sigma_q^2(0) > 0$.

Now, we have to take the limit as $K \to \infty$. Since, the v_K are bounded, the sums have finite fix number of terms and the a, b are constant, the important ingredients are the covariances. We split the domain of integration into two parts: on a tubular neighborhood of radio η of the diagonal $s = t$ and $s = t = s' = t'$ respectively and its complement.

Therefore, we have:

the integral close to the diagonal: we assume that $|t - s| < \eta, |t' - s'| < \eta, |t' - t| < \eta$ and $|s' - s| < \eta$.

The covariances are bounded, in absolute value, from above by constants, for instance

$$\left|cov\left(\overline{T}_K(s), \overline{\overline{T}'_K}(t)\right)\right| = \frac{1}{V_K(s)V_K(t)v_K(t)} \left|\partial_t r_K(s,t) - \frac{c'_K(2t)}{2V_K^2(t)} r_K(s,t)\right|$$

with $r_K(s,t) = \frac{1}{2}(c_K(\tau) + c_K(\sigma))$. At $\tau = 0$ we have $c_K(0) = 1$, $c'_K(0) = 0$ and $c''_K(0) = -\frac{(K+1)(2K+1)}{6K^2}$, thus, by continuity the terms involving τ are bounded. The remaining quantities are easily seen to be bounded by Lemma 5.6 and Inequalities (5.13) and (5.14).

Therefore, the integral is bounded by a constant times the volume of the tubular neighborhood of the diagonal. Such volume is proportional to $K\pi$.

In conclusion, the $1/(K\pi)^2$ is not compensated by the integral, so the upper bound for the integral in the formula for the norm of the contraction (restricted to the neighborhood of the diagonal) tends to zero.

the integral far from the diagonal: In the rest of the proof of the CLT, C stands for some constant which actual value is meaningless, but not depending on K,

We may assume that at least one of the following $|t-s|>\eta$, $|t'-s'|>\eta$, $|t'-t|>\eta$, $|s'-s|>\eta$ holds true.

By Inequalities (5.13) and (5.14), the (absolute value of the) covariances $\overline{r}$, $\widetilde{r}'$ and $\widetilde{r}''$ at x, y are bounded from above by $C(1/|x-y|+1/|x+y|)$ when $|x-y|>\eta$. Therefore, the product in the integrand of $\|g_q \otimes_n g_q\|_2^2$ is bounded by the product of $1/|t-s|+1/|t+s|$, $1/|t'-s'|+1/|t'+s'|$, $1/|s-s'|+1/|s+s'|$ and $1/|t-t'|+1/|t-t'|$. Each one of these factors appears in the bound if the distance between the corresponding variables is larger than η. Otherwise, we bound them by constant as in the previous case.

Furthermore, the exponents of the covariances involving s,t and s',t' is $n=1,\dots,q-1$, the sum of the exponents in the factors involving s,s' is $q-n=1,\dots q-1\geq 1$, and the sum of the exponents in the factors involving t,t' also is $q-n\geq 1$.

Let us consider one of the possible cases, the others are similar. Say that $n=1$, $|t-s|>\eta$, $|t'-s'|<\eta$, $|t'-t|<\eta$ and $|s'-s|<\eta$, call A the set of points verifying these inequalities. We bound the covariances involving the variables t',s'; t',t; and s',s in the integrand by constants. Then, the integrand is bounded by

$$
\begin{aligned}
C\iiiint_A \left[\frac{1}{\tau}+\frac{1}{\sigma}\right] ds dt ds' dt' \\
&= C\iiiint_A \frac{1}{\tau} d\tau dt ds' dt' + C\iiiint_A \frac{1}{\sigma} d\sigma dt ds' dt' \\
&\leq C\log(K\pi)\iiint_{A_{-1}} dt ds' dt'
\end{aligned}
$$

where $\tau=t-s$, $\sigma=t+s$ and $A_{-1}=\{(t,s',t') : |t'-s'|<\eta, |t'-t|<\eta, |s'-s|<\eta\}$. Now, the volume of A_{-1} is bounded by the volume of the tubular neighborhood of the diagonal of radius $\sqrt{3}\eta$, its volume is bounded by constant times $K\pi$, the result follows. Thus, the integral is bounded by a constant times $\log(K\pi)K\pi$, divided by $(K\pi)^2$, it tends to zero.

Putting together both parts we conclude that $\|g_q \otimes_n g_q\|_2^2 \to 0$ as $K\to\infty$ for $n=1,\dots,q-1$. Therefore, $I_q^{\overline{T}_K}$ converges in distribution to a Gaussian random variable as $K\to\infty$ for all q.

Asymptotic Gaussianity of the sum: Since, the $I_q^{\overline{T}_K}$'s are orthogonal, by Theorem 1 of Peccatti & Tudor [68] and Lemma 5.4, their Q-th partial sum converges in distribution, as K grows to infinity, to a Gaussian random variable with variance $\sum_1^Q var\left(I_q^X\left([0,K\pi]_{-\alpha}\right)\right)$, therefore

$$\frac{N_{\overline{T}_K} - \mathbb{E}\, N_{\overline{T}_K}}{\sqrt{K\pi}} = \sum_{q=1}^{\infty} I_q^{\overline{T}_K}$$

converges in distribution to a Gaussian random variable with variance $V^2 = \sum_{q=1}^{\infty} var\left(I_q^X\left([0,K\pi]_{-\alpha}\right)\right)$. (It is well known that if $\mu_n \to \mu_\infty$, $\sigma_n \to \sigma_\infty$, $X_n \sim N(\mu_n, \sigma_n^2)$ then $X_n \Rightarrow X_\infty$.) Finally, reasoning as in Lemma 5.1, note that $var\left(I_q^X\left([0,K\pi]_{-\alpha}\right)\right) = var\left(I_q^X\left([0,K\pi]\right)\right) = \sigma_q^2(0)$. This proves the Theorem.

5.4 Auxiliary computations

In the proof of Theorem 5.5 we use the following auxiliary results.

Lemma 5.5. *1. The second moment of the number of roots of $\overline{T}_K$ is finite, that is, $\mathbb{E}\,(N_{\overline{T}_K})^2 < \infty$.*

2. The second moment of the level crossings of $\overline{T}_K$ is continuous with respect to the level, that is, $\mathbb{E}\, N_u^2$ is continuous in u.

3. The approximation $N_{\overline{T}_K}^\eta$ converges in L^2 to $N_{\overline{T}_K}$.

4. The approximation $N_{\overline{T}_K}^\eta$ has a chaotic expansion.

Proof. 1. This is a consequence of Lemma 5.2, which bound the integral near the diagonal (the difficult part), or of Corollary 3.7 of Azaïs & Wschebor [12].

2. This is a consequence of Lemma 5.2 and Gaussianity, since it provides an integral expression involving the joint density of $\overline{T}_K$ at s and t and the conditional expectation of its derivatives with respect to its values at s and t. These quantities are continuous with respect to u and the necessary domination is given by part 1.

3. First, note that $N_{\overline{T}_K}^\eta \to N_{\overline{T}_K}$ almost surely since φ_η approximates the unity and $N_{\overline{T}_K}$ is almost surely constant.

   Therefore, it suffices to state the convergence of the second moment. Fatou's Lemma implies that

$$\mathbb{E}\, N_{\overline{T}_K} = \mathbb{E} \lim_\eta N_{\overline{T}_K}^\eta \leq \lim_\eta \mathbb{E}\, N_{\overline{T}_K}^\eta.$$

On the other hand, Area-Formula, see Federer [34] applied for $d = 1$, $B = [0, t]$, $g = \varphi_\eta$ and $f = \overline{T}_K \in C^1$), almost surely, permits us to write

$$N^\eta_{\overline{T}_K} = \int_{-\infty}^{\infty} N_{\overline{T}_K}(u)\varphi_\eta(u)du = \mathbb{E}_Z(N_{\overline{T}_K}(Z)),$$

where Z has density φ_η. Thus, applying Jensen's inequality in the inner expectation and Tonelli's Theorem, we have

$$\begin{aligned}\mathbb{E}\, N^2_{\overline{T}_K} &= \mathbb{E}\left((\mathbb{E}_Z(N_{\overline{T}_K}(Z)))^2\right) \leq \mathbb{E}\left(\mathbb{E}_Z(N^2_{\overline{T}_K}(Z))\right) \\ &= \int_{-\infty}^{\infty} \mathbb{E}\left(N^2_{\overline{T}_K}(u)\right)\varphi_\eta(u)du\end{aligned}$$

Since, φ_η approximates the unity, since $\mathbb{E}\, N^2_{\overline{T}_K}(u)$ is continuous in u (part 2 of this lemma), passing to the limit, with $\eta \to 0$, in the latter inequality gives the desired result.

4. Since $\overline{r}^{(11)}_K$ is bounded, it follows from Lemma 2 in Kratz & León [49] with minor changes.

□

Derivatives and covariances

Let us start computing the derivatives of V_K and V (the variances of T_K and T respectively), it is immediate from Formula (5.10) that

$$V'_K(t) = \frac{c'_K(2t)}{2V_K(t)}, \quad V'(t) = \frac{\mathrm{sc}'(2t)}{2V(t)} \tag{5.23}$$

Now let us compute the covariances between $\overline{T}_K$ and its derivatives. First, observe that

$$\overline{T}'_K(t) = \frac{1}{V_K(t)}\left(T'_K(t) - \frac{V'_K(t)}{V_K(t)}T_K(t)\right)$$

and

$$\overline{T}'(t) = \frac{1}{V(t)}\left(T'(t) - \frac{V'(t)}{V(t)}T(t)\right)$$

Therefore ($s < t$)

$$\begin{aligned}\mathbb{E}\,\overline{T}(s)\overline{T}'(t) &= \mathbb{E}\,\frac{T(s)}{V(s)}\left[\frac{1}{V(t)}\left(T'(t) - \frac{V'(t)}{V(t)}T(t)\right)\right] \\ &= \frac{1}{V(s)V(t)}\left[\partial_t r(s,t) - \frac{\mathrm{sc}'(2t)}{2V(t)^2}r(s,t)\right] \\ &= \frac{1}{\sqrt{1+\mathrm{sc}(2s)}\sqrt{1+\mathrm{sc}(2t)}}\left[\mathrm{sc}'(\tau) + \mathrm{sc}'(\sigma) - \frac{\mathrm{sc}'(2t)(\mathrm{sc}(\tau)+\mathrm{sc}(\sigma))}{1+\mathrm{sc}(2t)}\right]\end{aligned}$$

The covariance between $\overline{T}'(s)$ and $\overline{T}'(t)$ is

$$\begin{aligned}\mathbb{E}\,\overline{T}'(s)\overline{T}'(t) &= \frac{1}{V(s)V(t)}\mathbb{E}\left(T'(s)-\frac{V'(s)}{V(s)}T(s)\right)\left(T'(t)-\frac{V'(t)}{V(t)}T(t)\right)\\ &= \frac{1}{V(s)V(t)}\left[\partial_{st}r(s,t)-\frac{V'(t)}{V(t)}\partial_s r(s,t)-\frac{V'(s)}{V(s)}\partial_t r(s,t)+\frac{V'(s)V'(t)}{V(s)V(t)}r(s,t)\right]\end{aligned}$$

The variance of $\overline{T}'(t)$ is

$$\begin{aligned}&\frac{1}{V^2(t)}\mathbb{E}\left(T'(t)-\frac{V'(t)}{V(t)}T(t)\right)^2\\ &\qquad= \frac{1}{V^2(t)}\left[var(T'(t))-\frac{2V'(t)}{V(t)}\mathbb{E}\,T(t)T'(t)+V'(t)^2\right]\\ &\qquad\qquad= \frac{1}{V^2(t)}\left[var(T'(t))-V'(t)^2\right]\end{aligned}$$

where we take into account that $\mathbb{E}\,T(t)T'(t) = V'(t)V(t)$, which follows from a direct computation.

Therefore

$$v_K^2(t) = var(\overline{T}'(t)) = \frac{1}{1+\mathrm{sc}(2t)}\left[\mathrm{sc}''(2t)-\mathrm{sc}''(0)-\frac{\mathrm{sc}'(2t)^2}{1+\mathrm{sc}(2t)}\right] \tag{5.24}$$

Similar formulas hold true for $\overline{T}_K$ replacing sc by c_K.

Lemma 5.6. *There exists t_0 large enough, such that the variances of T_K, T and $\overline{T}'_K$, $\overline{T}'$ are bounded away from zero on the interval in $[t_0, K\pi - t_0]$.*

Proof. Let us start with the proof for T and $\overline{T}'$.

In the case of T, it suffices to prove that $\mathrm{sc}(x) > -2/\pi$. Observe that the critical points of sc are the roots of the equation $\mathrm{sc}(x) = \cos(x)$, then, it is easy to see that in the interval $[-\pi/2, \pi, 2]$ the only root of this equation is $x = 0$, which is a maximum since $\mathrm{sc}(0) = 1$. Thus, the minimum of sc lies outside this interval, so, as $|\mathrm{sc}(x)| \leq 1/|x|$ the minimum is greater than $-\pi/2$, thus the result follows.

Let us look now at the variance of $\overline{T}'(t)$ which is given in Equation (5.24). It is easy to see that

$$v_K^2(t) \geq \frac{1}{2}\left[\frac{1}{3}-\mathrm{sc}''(2t)-\frac{\mathrm{sc}'(2t)^2}{1+\mathrm{sc}(2t)}\right]$$

and from the Inequalities in (5.13), which hold since we are looking at compact intervals not containing 0, it is easy to see that the latter expression is bounded away from zero for large t.

Now, let us consider the variances of T_K and $\overline{T}'_K$. By Equation (37) in [37], we know that

$$c_K^{(j)}(2s) = \mathrm{sc}^{(j)}(2s) + O(1/N^j),$$

where $j = 0, 1, 2$ stands for functional value, first or second derivative respectively, from this asymptotic and the result for T and $\overline{T}'$, the result follows also for T_K and $\overline{T}'_K$. □

Chapter 6

A probabilistic approach to Bézout's Theorem

In this chapter we consider a random square system of complex polynomial equations and apply the classical multivariate Rice Formula to the problem of computing the number of zeros of such a system. In this sense, we give a probabilistic approach to the proof of Bézout's Theorem about the number of roots of a system of m complex variables and m equations with complex coefficients. Actually we give the proof in some particular cases as quadratic systems.

Bézout's Theorem is a generalization to the multidimensional case of the Fundamental Theorem of Algebra. It is worth to remark that it is a classic (well known) result. But the known proofs rely on very different techniques than ours, such as integral and algebraic geometry and elimination theory. For example, in Blum, Cucker, Shub & Smale [19] a proof is given relying on the Homotopy Method, which construct a continuous path of solutions departing from the solution of a particular simple known system and ending at the solution of the given system. One of the remarkable facts about this method is that it can be combined with Newton's Method in order to obtain a numeric algorithm to find the solution of the system. Our aim is to provide a new point of view for this result.

It is worth to say, that the study of random polynomials and random systems of polynomial equations is inspired by the average approach to the complexity of algorithms. In that respect, originally, people used to study the best and worse cases, but since the eighties and in particular in Smale [78], it was proposed a probabilistic analysis of complexity.

This problem was originally proposed by Mario Wschebor.

We now begin with some preliminaries, in particular we set the notations for the involved systems of equations and define the randomization we use for them,

then, we adapt Rice Formula for this context.

6.1 Introduction

Consider m polynomial equations on m complex variables $t_1, \ldots, t_m$ and complex coefficients $a_j^{(\ell)}$, where the index ℓ stands for the equation and the multi-index j stands for the term (monomial) in which the coefficient appears.

We denote the system in vector form $f = 0$, being $f = (f_1 \ldots, f_m)$. Then, we can write

$$f_\ell(t) = \sum_{\|j\| \leq d_\ell} a_j^{(\ell)} t^j, \quad \ell = 1, \ldots, m, \tag{6.1}$$

where d_ℓ is the degree of the polynomial f_ℓ, $j = (j_1 \ldots, j_m) \in \mathbb{N}^m$ is a multi-index of non-negative integers, $t = (t_1 \ldots, t_m) \in \mathbb{C}^m$, $\|j\| = \sum_{k=1}^m j_k$ and $t^j := \prod_{k=1}^m t_k^{j_k}$.

It is convenient to deal with the homogenized version of the system, which is denoted (for a while) by $f^{(0)}$. In order to get that version, we introduce an auxiliary complex variable t_0 and redefine the system f so that all the monomials have the same degree, that is, in the j-th term of the ℓ-th equation, t_0 has exponent $d_\ell - \|j\|$. In other words, we replace the original system given by Equation (6.1) by the new one

$$f_\ell^{(0)}(t) = \sum_{\|j\| = d_\ell} a_j^{(\ell)} t^j, \quad \ell = 1, \ldots, m,$$

where now $j = (j_0, j_1 \ldots, j_m) \in \mathbb{N}^{m+1}$ and $t = (t_0, t_1 \ldots, t_m) \in \mathbb{C}^{m+1}$.

Since all the monomials of $f_\ell^{(0)}$ have the same degree d_ℓ it follows that $f_\ell^{(0)}(\lambda t) = \lambda^{d_\ell} f_\ell^{(0)}(t)$ for all $\lambda \in \mathbb{C}$ and $t \in \mathbb{C}^{m+1}$.

Now, let us describe the relation between the roots of the original and the homogenized systems. Suppose that $s \in \mathbb{C}^m$ is a root of f, that is $f(s) = 0$, then $f^{(0)}(\lambda s) = 0$ for all $\lambda \in \mathbb{C}$. Conversely, if $t \in \mathbb{C}^{m+1}$ is a root of $f^{(0)}$, then, for $t_0 \neq 0$, so is $(1, \frac{t_1}{t_0}, \ldots, \frac{t_n}{t_0})$ and $\frac{1}{t_0}(t_1, \ldots, t_n)$ is a root of f. If $t_0 = 0$ we say that t is a root at infinity, see Blum et al. [19] for details. In conclusion, individual roots of the original system correspond to complex lines of solutions of the homogenized system.

From now on, we drop the super-index in $f^{(0)}$, there is no risk of confusion.

One of the main questions about polynomial systems of equations is to know its number of roots, we take the approach of randomizing the coefficients of the system and to study the random variable N defined as the number of roots of the system.

Now, we enunciate informally, the theorem that we consider in this chapter, after that, we define the concepts we need to precise and prove it.

Theorem 6.1. *Let f be a complex polynomial system with m variables and m equations. If the coefficients of f are randomized, then, the number of roots of f on $\mathbb{C}^m$ is almost surely equal to*

$$N = \mathcal{D}$$

being $\mathcal{D} = \prod_{\ell=1}^{m} d_\ell$ Bézout's number.

This theorem is weaker than Bézout's Theorem since it does not give information about the set where the equality fails to hold true.

As we mentioned at the beginning of this chapter, Bézout's Theorem is a well known result, nevertheless, its proof is based on very different arguments of algebraic and differential geometry, in particular it uses the Fundamental Theorem of Elimination. On this chapter, we pretend to give a different point of view to this problem to help in its understanding.

It is worth to recall that in the case of polynomials with real variables and coefficients, Shub & Smale [76] proved that the mean number of real roots of such a system is the square root of Bézout's number. After that, Azaïs and Wschebor [11], see also [12], gave a proof of that result relying on Rice Formula. Besides, Wschebor [80] gave some asymptotics for the variance as m tends to infinity, see also Azaïs & Wschebor [12, Section 12.1.2].

This work is inspired on these seminal papers.

6.2 Preliminaries

In this section we describe the randomization used for the coefficients and adapt Rice formula to the complex case. We start with some basic issues.

6.2.1 Algebraic Structure

In the sequel we identify the complex line $\mathbb{C}$ with the real plane $\mathbb{R}^2$ by the map

$$s = x + iy \mapsto (x, y) \tag{6.2}$$

This map induces an identification on the operations of these linear spaces, namely: for $\lambda = a + ib \in \mathbb{C}$, $s = x_s + iy_s$ and $t = x_t + iy_t$ the maps $(s, t) \mapsto s + t$

and $s \mapsto \lambda \cdot s$ may be seen, respectively, as

$$((x_s, y_s), (x_t, y_t)) \mapsto (x_s + x_t, y_s + y_t)$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

Similarly, $\mathbb{C}^{m+1}$ is identified with $\mathbb{R}^{2m+2}$ by

$$(s_0, \ldots, s_m) \mapsto (x_0, \ldots, x_m, y_0, \ldots, y_m)$$

where we have denoted $s_j = x_j + iy_j$, for $0 \leq j \leq m$. Briefly: $s \mapsto (x, y)$, $s \in \mathbb{C}^{m+1}$, $x, y \in \mathbb{R}^{m+1}$. In particular, the complex sphere in $\mathbb{C}^{m+1}$ is identified with the real sphere , S^{2m+1}, in $\mathbb{R}^{2m+2}$.

If $s \in \mathbb{C}^{m+1}$ is a complex vector with real and imaginary parts $x, y \in \mathbb{R}^{m+1}$ respectively then, for $\lambda \in \mathbb{C}$, the map $s \mapsto \lambda \cdot s$ is represented by the same formula of the case $m = 1$ if we let the 2×2 matrix act on $\mathbb{R}^{2m+2}$ by

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix}$$

where ax denotes the product of the scalar $a \in \mathbb{R}$ with the vector $x \in \mathbb{R}^{m+1}$ and so on.

Lemma 6.1. *With the identifications induced by the map (6.2), if $s = x_s + iy_s, t = x_t + iy_t \in \mathbb{C}^{m+1}$, we have*

$$\Re \langle s, t \rangle_{\mathbb{C}^{m+1}} = \left\langle \begin{pmatrix} x_s \\ y_s \end{pmatrix}, \begin{pmatrix} x_t \\ y_t \end{pmatrix} \right\rangle_{\mathbb{R}^{2m+2}},$$

where $\Re$ denotes the real part, $\langle \cdot, \cdot \rangle_{\mathbb{C}^{m+1}}$ is the usual Hermitian inner product in $\mathbb{C}^{m+1}$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}^{2m+2}}$ is the usual inner product in $\mathbb{R}^{2m+2}$.

Proof. For $m = 0$, we have

$$(x_s + iy_s)\overline{(x_t + iy_t)} = x_s x_t + y_s y_t + i(x_t y_s - x_s y_t),$$

the desired result follows taking real parts on both sides.

For $m \geq 1$, by definition $\langle s, t \rangle_{\mathbb{C}^{m+1}} = \sum_{k=0}^{m} s_k \bar{t}_k$, then, the result follows from the preceding one for each term and the additivity of the real part. □

Let us see how to interpret the action of the unitary group of $\mathbb{C}^{m+1}$ in $\mathbb{R}^{2m+2}$. Let $U \in \mathcal{U}(\mathbb{C}^{m+1})$ an unitary operator, then we define its counterpart $O \in \mathcal{O}(\mathbb{R}^{2m+2})$ by

$$O = \begin{pmatrix} U_r & -U_{im} \\ U_{im} & U_r \end{pmatrix},$$

where $U_r = \mathbb{R}e(U)$ and $U_{im} = Im(U)$ are the real and imaginary parts of the matrix U respectively. Indeed, let $s = x + iy$, hence

$$O\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} U_r & -U_{im} \\ U_{im} & U_r \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} U_r x - U_{im} y \\ U_{im} x + U_r y \end{pmatrix},$$

and on the other hand $(U_r + iU_{im})(x + iy) = U_r x - U_{im} y + i(U_{im} x + U_r y)$. Hence

$$U(x + iy) \mapsto O(x, y).$$

Namely, we obtain the same result by applying U and then changing to the real set up than by changing to the real set up and then applying O. As a consequence, we have

Lemma 6.2. *The operator family*

$$\mathcal{O}' = \left\{ O = \begin{pmatrix} U_r & -U_{im} \\ U_{im} & U_r \end{pmatrix} : U \in \mathcal{U}(\mathbb{C}^{m+1}) \right\}$$

acts transitively on $S^{2m+1} \subset \mathbb{R}^{2m+2}$ *by isometries.*

Proof. Let $s = (x, y)$ and $t = (x', y') \in S^{2m+1}$. Since $\mathcal{U}(\mathbb{C}^{m+1})$ acts transitively on the unit sphere on $\mathbb{C}^{m+1}$, there exists an unitary map $U \in \mathcal{U}(\mathbb{C}^{m+1})$ such that $U(s) = t$. Hence, by the arguments preceding this Lemma, the real part O of U maps s into t on S^{2m+1}.

Let us verify that $O \in \mathcal{O}(\mathbb{R}^{2m+2})$, by Lemma 6.1

$$\begin{aligned} \left\langle O\begin{pmatrix} x_s \\ y_s \end{pmatrix}, O\begin{pmatrix} x_t \\ y_t \end{pmatrix} \right\rangle &= \mathbb{R}e\left\langle Us, Ut \right\rangle_{\mathbb{C}^{m+1}} = \mathbb{R}e\left\langle s, t \right\rangle_{\mathbb{C}^{m+1}} \\ &= \left\langle \begin{pmatrix} x_s \\ y_s \end{pmatrix}, \begin{pmatrix} x_t \\ y_t \end{pmatrix} \right\rangle. \end{aligned}$$

This concludes the proof. □

To end the Preliminaries, let us say how to relate the derivatives (Jacobian matrices) of real and complex expressions of an holomorphic function $f : \mathbb{C}^{m+1} \to \mathbb{C}^{m+1}$.

Denote $f = (f_1, \ldots, f_{m+1})$, and $f_k = u_k + iv_k$ the real and imaginary parts of f_k. The complex derivative f' is represented by the $(m+1) \times (m+1)$ complex matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} & \cdots & \frac{\partial f_1}{\partial z_{m+1}} \\ \\ \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} & \cdots & \frac{\partial f_2}{\partial z_{m+1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_{m+1}}{\partial z_1} & \frac{\partial f_{m+1}}{\partial z_2} & \cdots & \frac{\partial f_{m+1}}{\partial z_{m+1}} \end{pmatrix}$$

and the real counterpart, that is the differential, Df, of f viewed as a function from $\mathbb{R}^{2m+2}$ into $\mathbb{R}^{2m+2}$, is represented by the $(2m+2)\times(2m+2)$ real matrix

$$J(f) = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_{m+1}} & \frac{\partial u_1}{\partial y_1} & \cdots & \frac{\partial u_1}{\partial y_{m+1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_{m+1}}{\partial x_1} & \cdots & \frac{\partial u_{m+1}}{\partial x_{m+1}} & \frac{\partial u_{m+1}}{\partial y_1} & \cdots & \frac{\partial u_{m+1}}{\partial y_{m+1}} \\ \\ \frac{\partial v_1}{\partial x_1} & \cdots & \frac{\partial v_1}{\partial x_{m+1}} & \frac{\partial v_1}{\partial y_1} & \cdots & \frac{\partial v_1}{\partial y_{m+1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial v_{m+1}}{\partial x_1} & \cdots & \frac{\partial v_{m+1}}{\partial x_{m+1}} & \frac{\partial v_{m+1}}{\partial y_1} & \cdots & \frac{\partial v_{m+1}}{\partial y_{m+1}} \end{pmatrix}$$

with $z_k = x_k + iy_k$.

The following Lemma is a well known result from Complex Analysis, see for instance Range [72, Lm. 2.1].

Lemma 6.3. *If $f : \mathbb{C}^{m+1} \to \mathbb{C}^{m+1}$ is an holomorphic function, then*

$$\det(J(f)(z)) = |\det f'(z)|^2$$

for all z in its domain of holomorphy.

6.2.2 Complex random variables

We turn now to the Gaussian distribution on the complex plane, actually, we only need a simplified version which can be expressed in terms of t (rather than in terms of t and $\bar{t}$ as in the general case, see Ben Hough et al. [38] for the simplified version or Picinbono [70] for the complete one).

A complex random variable $T : \Omega \to \mathbb{C}$ can be written in the form $T = X + iY$, where X and Y are real random variables, called the real and imaginary parts of T respectively. We say that $T = X + iY$ has the complex Gaussian distribution if the random vector (X, Y) has joint real Gaussian distribution in $\mathbb{R}^2$.

More precisely, $T = X + iY$ has standard complex Gaussian distribution, written $T \sim \mathcal{N}_{\mathbb{C}}(0, 1)$, see Ben Hough et al. [38], if its real and imaginary parts are (real) independent centered Gaussian random variables with variance $1/2$, that is, $(X, Y) \sim \mathcal{N}(0, (1/2)\cdot \mathbf{I})$, being $\mathbf{I}$ the identity matrix. Thus, its density with respect to the Lebesgue measure on the plane is

$$p(t) = \frac{1}{\pi} e^{-|t|^2}. \tag{6.3}$$

Generally, the Gaussian distribution with mean $\mu \in \mathbb{C}$ and variance $\sigma^2 > 0$ has the density

$$p(s) = \frac{1}{\pi\sigma} e^{-\frac{|s-\mu|^2}{\sigma^2}}. \tag{6.4}$$

It is easy to check that $\mathbb{E}\, T := \mathbb{E}\, X + i\mathbb{E}\, Y = \mu$, $\mathbb{E}\left[(T-\mu)\overline{(T-\mu)}\right] = \sigma^2$ and $\mathbb{E}\left[(T-\mu)(T-\mu)\right] = 0$.

Analogously, we say that the complex random vector $T = X + iY$ in $\mathbb{C}^{m+1}$ has the complex Gaussian distribution if the corresponding real vector (X, Y) in $\mathbb{R}^{2m+2}$ has Gaussian distribution.

We say that the complex random variables S, T are independent if their respective real and imaginary parts are so. The following Lemma gives a practical criterion to check the independence of two complex Gaussian random variables.

Lemma 6.4. *Let S, T be complex Gaussian random variables with $\mu_S := \mathbb{E}\, S$ and $\mu_T := \mathbb{E}\, T$, then, S and T are independent if and only if*

$$\mathbb{E}\left[(S-\mu_S)\overline{(T-\mu_T)}\right] = 0, \quad \mathbb{E}\left[(S-\mu_S)(T-\mu_T)\right] = 0$$

Remark 6.1. *Actually, more generally we can say that the complex Gaussian distribution is determined by two parameters, namely, its mean and its variance matrix. In other words, it is a second order distribution.*

Proof. Without lose of generality, assume that S and T are centered. Denote by S_r, T_r and S_{im}, T_{im} the real and imaginary parts of S and T respectively. Therefore,

$$\begin{aligned}
\mathbb{E}\, ST &= (\mathbb{E}\, S_r T_r - \mathbb{E}\, S_{im} T_{im}) + i(\mathbb{E}\, S_r T_{im} + \mathbb{E}\, S_{im} T_r) \\
\mathbb{E}\, S\overline{T} &= (\mathbb{E}\, S_r T_r + \mathbb{E}\, S_{im} T_{im}) + i(-\mathbb{E}\, S_r T_{im} + \mathbb{E}\, S_{im} T_r)
\end{aligned}$$

A bit of linear algebra establish the result. □

The covariance between two complex random variables is defined by

$$cov(S, T) = \mathbb{E}\left[(S - \mathbb{E}\, S)\overline{(T - \mathbb{E}\, T)}\right]$$

On our case, the expectation of the product of two centered random variables vanishes since we assume (as we can do so by Corollary 6.1) that the real and imaginary parts of each one are independent. Therefore, the density function of a centered Gaussian vector with variance matrix Σ is given by

$$p(s) = \frac{1}{(2\pi)^m \sqrt{|\det(\Sigma_r)|}} \exp\left\{-\frac{1}{2}(s-\mu)^t \Sigma^{-1}(s-\mu)\right\} \tag{6.5}$$

6.3 Weyl distribution

We are now ready to introduce the randomization that we use for the coefficients of the system f of random complex polynomial equations, namely, Weyl distribution.

This distribution was used in 1992 by Shub & Smale [76] to get the first important result on the number of roots of systems of equations, later it was extended to some extent, see Armentano [6] and references therein.

As we shall see, this law has the key property of being invariant under isometries, in Kostlan [47] are classified all Gaussian distributions over the coefficients of the system that yield this geometric property.

We say that the homogeneous polynomial system $f = (f_1, \dots, f_m)$ on $m+1$ complex variables has the Weyl distribution if the coefficients $a_j^{(\ell)}$ are independent complex centered Gaussian random variables with variances

$$\binom{d_\ell}{j} = \frac{d_\ell}{j_0! \dots j_m!}.$$

That is, $a_j^{(\ell)} \sim \mathcal{N}_{\mathbb{C}}\left(0, \binom{d_\ell}{j}\right)$. It is immediate that

$$\mathbb{E}\left(a_j^{(\ell)}\right)^2 = 0, \qquad \mathbb{E}\, a_j^{(\ell)} \overline{a_j^{(\ell)}} = \binom{d_\ell}{j}. \tag{6.6}$$

The following Lemma contains the first remarkable properties of the Weyl distribution, in particular, the invariance under the action of the unitary group.

Lemma 6.5. *Let f be a system with the Weyl distribution, then, $f = (f(s) : s \in \mathbb{C}^{m+1})$ is a centered complex Gaussian process such that for every $s, t \in \mathbb{C}^{m+1}$*

$$\begin{aligned} \mathbb{E}\, f_\ell(s)\overline{f_\ell(t)} &= \langle s, t\rangle_{\mathbb{C}^{m+1}}^{d_\ell}, \\ \mathbb{E}\, f_\ell(s) f_\ell(t) &= 0. \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the usual Hermitian inner product in $\mathbb{C}^{m+1}$.

Proof. First observe that f is a centered complex Gaussian process since it is a linear combination of the centered complex Gaussian random variables $a_j^{(\ell)}$, for $\ell = 1, \dots, m$ and $j \in \mathbb{N}^{m+1}$.

Let us compute its covariance and relation (see Picinbono [70]). For the sake

of simplicity we omit the sub-index.

$$\begin{aligned}\mathbb{E}\left(f(s)\overline{f(t)}\right) &= \mathbb{E}\left[\sum_{\|j\|=d} a_j s^j \cdot \overline{\sum_{\|j'\|=d} a_{j'} t^{j'}}\right] \\ &= \sum_{\|j\|=d}\sum_{\|j'\|=d} \mathbb{E}\,(a_j\overline{a_{j'}}) s^j \overline{t^{j'}}.\end{aligned}$$

Since the a_j are independent and centered, for $j \neq j'$ we have $\mathbb{E}\,(a_j\overline{a_{j'}}) = \mathbb{E}\,a_j \mathbb{E}\,\overline{a_{j'}} = 0$. If $j = j'$, from Equation (6.6) we have $\mathbb{E}\,(|a_j|^2) = \binom{d}{j}$, hence

$$\begin{aligned}\mathbb{E}\left(f(s)\overline{f(t)}\right) &= \sum_{\|j\|=d} \mathbb{E}\,(|a_j|^2) s^j t^j = \sum_{\|j\|=d} \binom{d}{j} s^j t^j \\ &= \langle s, t\rangle^d_{\mathbb{C}^{m+1}},\end{aligned}$$

where we use Newton's binomial theorem.

For the second assertion of this lemma note that

$$\mathbb{E}\, f(s)f(t) = \sum_{\|j\|=d} \mathbb{E}\,(a_j^2) s^j t^j.$$

Then, the second assertion follows from Equation (6.6). □

As consequence we can establish the following key fact.

Corollary 6.1. *Let f be a complex Weyl polynomial system, denote $f(t) = u(t) + iv(t)$, being $u(t)$ and $v(t)$ the real and imaginary parts of f respectively, then, for each t, the random variables $u(t)$ and $v(t)$ are independent, further, they have real centered Gaussian law with variance $\|t\|^2_{\mathbb{C}^{m+1}}/2$.*

Proof. Using the previous Lemma we have

$$\begin{aligned}0 = \mathbb{E}\, f(s)f(t) &= \mathbb{E}\,(u(s) + iv(s))\,(u(t) + iv(t)) \\ &= \mathbb{E}\,(u(s)u(t) - v(s)v(t)) + i\mathbb{E}\,(v(s)u(t) + u(s)v(t))\end{aligned}$$

Therefore, from the real part we obtain

$$\mathbb{E}\, u(s)u(t) = \mathbb{E}\, v(s)v(t),$$

since the real and imaginary part of f are centered (real) Gaussian processes and last equation show that their covariances coincide, we conclude that the processes u and v have the same law. In particular, since $|f|^2 = u^2 + v^2$ the variance of $u(t)$ and $v(t)$ is $\mathbb{E}\,(|f|^2)/2$.

Furthermore, for $s = t$, since the imaginary part also vanish we have in addition

$$\mathbb{E}\, u(t)v(t) = -\mathbb{E}\, u(t)v(t).$$

It follows that $u(t)$ and $v(t)$ are independent for each t. □

6.4 Toward a probabilistic proof of Bézout's Theorem

We enunciate (the probabilistic version of) Bézout's Theorem. Recall that the Bézout's number $\mathcal{D}$ of the system is defined as the product of the degrees of its equations.

Theorem 6.2 (Bézout's Theorem)**.** *If the random, complex polynomial system f has the Weyl distribution, then, almost surely, it has $\mathcal{D}$ roots counted without multiplicity.*

We include the complete proof for the cases of one equation and one variable $m = 1$ (the Fundamental Theorem of Algebra), and the case of quadratic systems $m > 1, d_\ell = 2$. In the general case, the computations of the second moment get quite involved and we can not preform them.

6.4.1 The roots of f on the projective space and on the sphere

Let $f = (f_1, \ldots, f_m)$ be a complex homogeneous system on m equations and $m+1$ complex variables.

As mentioned above, if $t_0 \in \mathbb{C}^{m+1}$ is a root of the system, that is $f(t_0) = 0$, then $f_\ell(\lambda z_0) = \lambda^{d_\ell} f_\ell(z_0) = 0$ for all $\lambda \in \mathbb{C}$, hence, the roots of f are complex lines through the origin in $\mathbb{C}^{m+1}$.

Clearly, this fact suggests to work on the projective space $\mathbb{P}(\mathbb{C}^{m+1})$ or on the complex sphere in $\mathbb{C}^{m+1}$, $S^{2m+1} = \{t \in \mathbb{C}^{m+1} : \|t\| = 1\}$.

On the sphere S^{2m+1}, if $f(t_0) = 0$, then, $f(\lambda t_0) = 0$ for all $\lambda \in \mathbb{C}$ of modulus one, therefore, the roots of f are complex circumferences (i.e: one-dimensional spheres) in S^{2m+1}.

Let us express these facts in terms of $\mathbb{R}^{2m+2}$. Each projective root $t_0 = x_0 + iy_0 \in \mathbb{P}(\mathbb{C}^{2m+1})$ is associated with the set of roots

$$\left\{ e^{i\theta} \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} : \theta \in [0, 2\pi) \right\}$$

on the sphere S^{2m+1} in $\mathbb{R}^{2m+1}$, these sets are circumferences (dimension one), then, the length of each one of them is 2π.

Besides, the intersection of each pair of these circumferences has almost surely zero Lebesgue measure, see Blum et al. [19]. Therefore, the number of complex

projective roots of f is equal to $1/(2\pi)$ times the geometric measure ($\sigma(d\cdot)$) of the union of the images of the roots through the identification via the map (6.2) with $\mathbb{R}^{2m+2}$. That is

$$N = \frac{1}{2\pi}\sigma_1^{-1}(f^{-1}(0) \cap S^{2m+1}). \tag{6.7}$$

The proof of Bézout's Theorem is divided into two parts, the first one is to prove that the mean number of roots equals Bézout's number, that is $\mathbb{E}\, N = \mathcal{D}$, and the second one is to prove that the variance of the number of roots vanish, that is $var(N) = 0$. We turn into these computations.

6.4.2 Computation of the first moment, the general case

These computations are general, that is, we obtain the mean number of roots of a Weyl system for all m, and d.

We use the identification induced by the map (6.2) described above, in order to write the Rice Formula for the number of roots of the system f. Denote by σ_{2m+1} the geometric $2m+1$-dimensional measure on the sphere S^{2m+1}. By Equation (6.7)

$$\begin{aligned}\mathbb{E}\, N &= \frac{1}{2\pi}\sigma_1(f^{-1}(0)) \\ &= \frac{1}{2\pi}\int_{S^{2m+1}} \mathbb{E}\left[|\det(Df(t)\cdot Df(t)')|^{1/2}\big|f(t)=0\right] p_{f(t)}(0)\sigma_{2m+1}(dt).\end{aligned}$$

Here Df stands for the real derivative of $f(x,y)$ along the manifold S^{2m+1}, and $p_{f(s)}$ stands for the real density of $f(x,y)$.

Nevertheless, as we shall see soon, sometimes is convenient to use the complex expressions of these quantities rather than the real ones in order to simplify the computations. But before passing to that task, we can also simplify the integral due to the invariance of the Weyl law.

In fact, since

$$\mathcal{O}' = \left\{ O = \begin{pmatrix} U_r & -U_{im} \\ U_{im} & U_r \end{pmatrix} : U \in \mathcal{U}(\mathbb{C}^{m+1}) \right\}$$

acts transitively on S^{2m+1} by isometries, it follows that the random vectors $f(t) : t \in S^{2m+1}$ have the same distribution and, hence, the integrand is constant, then

$$\mathbb{E}\, N = \frac{\sigma_{2m+1}(S^{2m+1})}{2\pi}\mathbb{E}\left[|\det(Df(t_0)\cdot Df(t_0)')|^{1/2}\big|f(t_0)=0\right] p_{f(t_0)}(0) \tag{6.8}$$

where t_0 is, say, the East Pole $(e_0, 0)$ of S^{2m+1}.

In order to clarify the notation, let us recall that the first component of the vectors in $\mathbb{R}^{2m+2}$ stands for the real part and the second for the imaginary part of the corresponding original point in $\mathbb{C}^{m+1}$. That is, the point $t_0 = (e_0, 0)$ with $e_0, 0 \in \mathbb{R}^{m+1}$ correspond, through the map (6.2), to the point $e_0^{\mathbb{C}} := e_0 + 0\,i \in \mathbb{C}^{m+1}$. In the rest of the proof we drop the super-index $\mathbb{C}$, but the vectors are all in $\mathbb{C}^{m+1}$.

Now, we compute separately the ingredients in Equation (6.8).

The density of $f(e_0)$ and the volume of the sphere

It is well known that the volume of the sphere is given by

$$\sigma_{2m+1}(S^{2m+1}) = \frac{2\pi^{(2m+1+1)/2}}{\Gamma((2m+1+1)/2)} = \frac{2\pi^{m+1}}{\Gamma(m+1)}$$

We turn now to the computation of the density $f(e_0)$. We proceed using the complex formula, but the same result is obtained via the real one.

By Lemma 6.5 and the fact that $\|e_0\|_{\mathbb{C}^{m+1}} = 1$, each one of the polynomials f_ℓ has standard complex Gaussian distribution. Further, the polynomials f_ℓ are independent for different values of ℓ, therefore

$$p_{f(e_0)}(0) = \frac{1}{\pi^m}.$$

Determinant: Since, for each ℓ, $f_\ell(t)$ has constant variance along the sphere, it follows that $f_\ell(t)$ and $f'_\ell(t)$ are independent for each t, see Corollary 6.1, therefore, f and Df are also independent for each t and we can drop the condition in the conditional expectation appearing in Equation (6.8), that is

$$\mathbb{E}\left[|\det(Df(e_0)\cdot Df(e_0)')|^{1/2}\middle| f(e_0) = 0\right] = \mathbb{E}\,|\det(Df(e_0)\cdot Df(e_0)')|^{1/2}.$$

Now, since this is an ordinary expectation, we can compute it directly. We need first to know the structure of the matrix $Df(e_0)$, this is pursued in the following lemma.

Lemma 6.6. *We have*

$$\mathbb{E}\,|\det(Df(e_0)\cdot Df(e_0)')|^{1/2} = \mathcal{D}\,\mathbb{E}\,|\det(G_m)|^2,$$

where G_m is an $m\times m$ matrix whose entries are i.i.d. standard, complex, Gaussian random variables and $\mathcal{D}$ is Bézout's number.

Proof. Let us express the derivatives in their complex versions.

In the first place, by Lemma 6.3, we have

$$|\det(Df(e_0)\cdot Df(e_0)')|^{1/2} = |\det f'(e_0)|$$

The selection of e_0 as the East Pole, permits us to write the matrix $f'(e_0)$ restricted to the tangent space $T_{e_0}S^{2m+1}$ in terms of the canonical basis $\{e_1,\dots,e_m\}$ of $T_{e_0}S^{2m+1}$. By Lemma 6.5, we know that $(f_\ell(s), f_\ell(t))$ is centered Gaussian with covariance $\langle s,t\rangle^{d_\ell}$. Then, it follows that the derivatives $\partial_k f_\ell(e_0)$ are centered complex Gaussian random variables, and taking derivatives of the covariances of f_ℓ we have

$$\mathbb{E}\,\partial_h f_\ell(s)\partial_k f_\ell(t) = d_\ell\,\langle s,t\rangle^{d_\ell-1}\,\delta_{hk} + d_\ell(d_\ell-1)\,\langle s,t\rangle^{d_\ell-2}\,t_k,$$

where δ is Kronecker delta and t_k is the k-th component of the vector t. Evaluating at $s=t=e_0$

$$\mathbb{E}\,\partial_h f_\ell(s)\partial_k f_\ell(t) = d_\ell\delta_{hk},$$

We conclude that the entries of $f'(e_0)$ are independent, centered complex Gaussian random variables with variances d_ℓ.

Therefore, by the properties of the Gaussian distribution, we can write $\partial_k f_\ell(e_0) = \sqrt{d_\ell}g_{k\ell}$, being $g_{k\ell}$ a standard complex Gaussian random variable independent from the rest. The properties of determinants do the rest. The lemma follows. □

In the following lemma we compute the determinant of the matrix G_m, at this point it is evident the simplification in the computations, at the end of the chapter we include another proof of this facts via the real arguments.

Lemma 6.7. *Let G_m be an $m\times m$ matrix with i.i.d. standard complex Gaussian entries, then*

$$\mathbb{E}\,|\det(G_m)|^2 = m!.$$

Proof. Let us compute $\mathbb{E}\,|\det(G_m)|^2$ by its very definition, for that, denote by S_m the symmetric group (on m elements). Then

$$\begin{aligned}\mathbb{E}\,|\det(G_m)|^2 &= \mathbb{E}\,\det(G_m)\det(\overline{G_m})\\ &= \sum_{\pi,\pi'\in S_m}(-1)^{\pi}(-1)^{\pi'}\mathbb{E}\,(g_{1\pi(1)}\dots g_{m\pi(m)}g_{1\pi'(1)}\dots g_{m\pi'(m)})\\ &= \sum_{\pi\in S_m}\mathbb{E}\,|g_{1\pi(1)}|^2\dots\mathbb{E}\,|g_{m\pi(m)}|^2 = m!\end{aligned}$$

In the second equality we use the independence and, in the last one, that $\mathbb{E}\,|g_{k\pi(k)}|^2 = 1$ for all k and that the cardinality of S_m is $m!$. □

Therefore, putting all this together, we have

$$\mathbb{E}\, N = \frac{1}{2\pi}\sigma_{2m+1}(S^{2m+1})\mathcal{D}m!\frac{1}{\pi^m} = \frac{1}{2\pi}\frac{2\pi^{m+1}}{\Gamma(m+1)}\mathcal{D}m!\frac{1}{\pi^m} = \mathcal{D}$$

as claimed.

6.4.3 Computation of the second moment

We compute now $\mathbb{E}\, N^2$, with N the number of projective roots of the system f. As in the previous computations, on S^{2m+1} each of these roots corresponds to a circumference, hence, $\sigma_1(f^{-1}(0)) = 2\pi N$.

Next lemma gives the formula for the second moment, note that the diagonal is relevant.

Lemma 6.8. *The second moment of the number of roots verifies*

$$\begin{aligned}4\pi^2\mathbb{E}\,(N^2) - 4\pi^2\mathcal{D}& \\ = \iint_{S^{2m+1}\times S^{2m+1}} \mathbb{E}\,&\left[|\det(f'(s))|^2|\det(f'(t))|^2 \mid f(s) = f(t) = 0\right] \\ &\cdot p_{f(s),f(t)}(0,0)dsdt,\end{aligned}$$

Proof. Following Azais & Wschebor [12], let $F : S^{2m+1} \times S^{2m+1} \to \mathbb{R}^{2m}$ be the map given by $F(s,t) = (f(s), f(t))$ and let $\Delta_\delta \subset S^{2m+1}\times S^{2m+1}$ be the set defined by $\Delta_\delta = \{(s,t) \in S^{2m+1} \times S^{2m+1} : \|s-t\| > \delta\}$. Then, applying Rice Formula for the geometric measure of $F^{-1}(0,0)$ we get:

$$\begin{aligned}\mathbb{E}\,\sigma_2(F^{-1}(0,0)\cap\Delta_\delta) = \iint_{\Delta_\delta} \mathbb{E}\,&\left[|\det(f'(s))|^2|\det(f'(t))|^2 \mid f(s) = f(t) = 0\right] \\ &\cdot p_{f(s),f(t)}(0,0)dsdt.\end{aligned}$$

Taking limit $\delta \downarrow 0$ we observe that

$$\mathbb{E}\,(\sigma_2(F^{-1}(0,0)\cap\Delta_\delta)) \uparrow \mathbb{E}\,(\sigma_2(F^{-1}(0,0))) - \mathbb{E}\,(\sigma_2(F^{-1}(0,0)\cap\Delta))$$

where $\Delta = S^{2m+1}\times S^{2m+1} - \Delta_0$ is the diagonal set.

Now, $F^{-1}(0,0) = \{(s,t) : f(s) = f(t) = 0\} \subset S^{2m+1}\times S^{2m+1}$, hence, since the roots are circumferences in S^{2m+1}, each pair of roots $s_h, t_k : h,k = 1,\dots,N$ add $(2\pi)^2$ to the geometric measure of $F^{-1}(0,0)$, thus, $\sigma_2(F^{-1}(0,0)) = (2\pi)^2N^2$. Similarly, each root $s_k : k = 1,\dots,N$ add $(2\pi)^2$ to the geometric measure of $F^{-1}((0,0)\cap\Delta)$, thus, $\sigma_2(F^{-1}(0,0)) = (2\pi)^2N$. Putting this together

$$\lim_{\delta\downarrow 0}\mathbb{E}\,(\sigma_1(F^{-1}(0,0)\cap\Delta_\delta)) = 4\pi^2\mathbb{E}\,(N^2) - 4\pi^2\mathcal{D}$$

Moreover, since Δ_0 has zero Lebesgue measure on $S^{2m+1} \times S^{2m+1}$ we conclude the lemma. □

We can simplify the integral in the same manner as we did for the first moment, but on this case we can fix just one point, not both. Let U be an unitary map on $\mathbb{C}^{m+1}$, then $(f(s), f(t))$ and $(f(Us), f(Ut))$ have the same distribution. Therefore, we can fix s, say at e_0, in the integral, hence

$$4\pi^2\mathbb{E}\,(N^2) - 4\pi^2\mathcal{D} = vol(S^{2m+1}) \cdot \int_{S^{2m+1}} \mathbb{E}\,\left[|\det(f'(e_0))|^2 |\det(f'(t))|^2 \mid f(e_0) = f(t) = 0\right] p_{f(e_0),f(t)}(0,0)dt.$$

Let us compute the ingredients of this formula separately.

Joint density: Since the different rows of the system are independent we have

$$p_{f(s),f(t)}(0,0) = \prod_{\ell=1}^{m} p_{f_\ell(s),f_\ell(t)}(0,0).$$

Routine manipulations of determinants show that we can use the complex covariance matrix, more precisely that $\det \Sigma_r = |\det \Sigma_c|^2$, with Σ_r and Σ_c the 4×4 real covariance matrix of u_k, v_k and the 2×2 complex covariance matrix of f_k $(k = 1, 2)$ respectively.

Furthermore, the covariance matrix of $(f_\ell(s), f_\ell(t))$ is, by Lemma 6.5, $\Sigma = \begin{bmatrix} 1 & \langle s,t\rangle^{d_\ell} \\ \langle t,s\rangle^{d_\ell} & 1 \end{bmatrix}$. Therefore $p_{f_\ell(s),f_\ell(t)}(0,0) = \frac{1}{\pi^2(1-|\langle s,t\rangle|^{2d_\ell})}$. Hence

$$p_{f(s),f(t)}(0,0) = \frac{1}{\pi^{2m}} \prod_{\ell=1}^{m} \frac{1}{1 - |\langle s,t\rangle|^{2d_\ell}}.$$

Conditional Expectation: In order to deal with an ordinary expectation rather than with a conditional one we proceed by Gaussian Regression, see Appendix A in page 131.

Regression of $f'(t)$ on $f(s) = f(t) = 0$

Choose $\alpha_{t\ell}, \beta_{t\ell}$ such that $\partial_t f(t) - \alpha f(t) - \beta f(s)$ is independent of $f(s), f(t)$. That is, α, β are the solution of the system:

$$\begin{cases} \alpha + \langle s,t\rangle^{d_\ell} \beta & = 0 \\ \langle t,s\rangle^{d_\ell} \alpha + \beta & = d_\ell \langle t,s\rangle^{d_\ell - 1} \langle s, v_t\rangle \end{cases}$$

Then

$$\alpha_{t\ell} = -\langle s,t\rangle^{d_\ell} \beta_{t\ell} \qquad \beta_{t\ell} = d_\ell \frac{\langle t,s\rangle^{d_\ell - 1} \langle s, v_t\rangle}{1 - |\langle s,t\rangle|^{2d_\ell}}.$$

The remaining $\alpha_{k\ell}, \beta_{k\ell}$ $(k \geq 2)$ vanish.

REGRESSION OF $f'(s)$ ON $f(s) = f(t) = 0$

The same arguments show that

$$\alpha_{1\ell} = -\langle t, s\rangle^{d_\ell} \beta_{1\ell} \qquad \beta_{1\ell} = d_\ell \frac{\langle s.t\rangle^{d_\ell - 1} \langle t, v_s\rangle}{1 - |\langle s, t\rangle|^{2d_\ell}}.$$

The remaining $\alpha_{k\ell}, \beta_{k\ell}$ $(k \geq 2)$ vanish.

In conclusion

$$\mathbb{E}\left[|\det(f'(s))|^2 |\det(f'(t))|^2 \mid f(s) = f(t) = 0\right] = \mathbb{E}\left[|\det(M(s))|^2 |\det(M(t))|^2\right], \tag{6.9}$$

where $M(s) = (\zeta^s_{\ell k})_{\ell k}$ and $M(t) = (\zeta^t_{\ell k})_{\ell k}$ are matrices with independent entries such that

$$\begin{aligned} \mathbb{E}\,\zeta^s_{\ell k}\overline{\zeta^s_{\ell k}} = \mathbb{E}\,\zeta^t_{\ell k}\overline{\zeta^t_{\ell k}} &= \begin{cases} d_\ell^2 & k \neq 1 \\ d_\ell^2 \sigma_\ell^2 & k = 1 \end{cases} \\ \mathbb{E}\,\zeta^s_{\ell k}\overline{\zeta^t_{\ell k}} &= \begin{cases} d_\ell^2 \langle s, t\rangle^{d_\ell - 1} & k \neq 1 \\ d_\ell^2 \tau_\ell & k = 1 \end{cases} \end{aligned}$$

where

$$\begin{aligned} \sigma_\ell^2 &= 1 - \frac{d_\ell |\langle s, t\rangle|^{2d_\ell - 2}}{1 + |\langle s, t\rangle|^2 + \cdots + |\langle s, t\rangle|^{2d_\ell - 2}} \\ \tau_\ell &= \langle s, t\rangle^{d_\ell - 2} \left[1 - \frac{d_\ell}{1 + |\langle s, t\rangle|^2 + \cdots + |\langle s, t\rangle|^{2d_\ell - 2}}\right] \end{aligned}$$

See the Auxiliary results.

The computations made above are general, they are valid for every m and d_ℓ. But, the computation of the determinant happens to be very difficult in that generality. Therefore, in the rest of the chapter we concentrate in some particular cases that allow to compute it.

Case $m = 1$

In this case we have that the conditional expectation is $\mathbb{E}\,(|\zeta|^2 |\zeta'|^2)$, where ζ, ζ' are complex centered Gaussian random variables with variance σ^2 and covariance τ. Applying Lemma 6.9 for $S = \zeta/\sigma$, $T = \zeta'/\sigma$ we deduce that

$$\mathbb{E}\,(|\zeta|^2 |\zeta'|^2) = \sigma^4 + |\tau|^2.$$

Thus, by Lemma 6.8, we have

$$\begin{aligned} 4\pi^2\mathbb{E}\,\mathcal{N}^2 - 4\pi^2\mathcal{D} &= \int_{S^3\times S^3} \frac{\sigma^4 + |\tau|^2}{\pi^2(1 - |\langle s,t\rangle|^{2d})} ds dt \\ &= \frac{vol(S^3)}{\pi^2}\int_{S^3} \frac{\sigma^4 + |\tau|^2}{(1 - |\langle e_0,t\rangle|^{2d})} dt \end{aligned}$$

Here, as points of S^3, $e_0 = (1,0,0,0)$ and $t = (t_0,t_1,t_2,t_3)$, corresponding to $e_0 = (1,0)$ and $t = (t_0 + it_1, t_2 + it_3) \in \mathbb{C}^2$.

Observe that the integrand depends only on the modulus of the Hermitian inner product of e_0 and t, so we may apply the Co-Area Formula, see for instance Blum et al. [19, Ch.13 Th. 4], for the map $\psi : S^3 \to \mathbf{D}$, being $\mathbf{D}$ the unitary disc in $\mathbb{C}$, such that $(e_0,t) \mapsto \langle e_0,t\rangle$. Then, the Normal Jacobian is $\sqrt{1-|x|^2}$, therefore

$$\begin{aligned} 4\pi^2\mathbb{E}\,\mathcal{N}^2 - 4\pi^2\mathcal{D} &= \frac{vol(S^3)}{\pi^2}\int_{\mathbf{D}} \frac{\sigma^4(x) + |\tau(x)|^2}{1-|x|^{2d}} dx \int_{S\left(\sqrt{1-|x|^2}\right)} \frac{1}{\sqrt{1-|x|^2}} d\theta \\ &= \frac{vol(S^3)vol(S^1)}{\pi^2}\int_{\mathbf{D}} \frac{\sigma^4 + |\tau|^2}{1-|x|^{2d}} dx, \end{aligned}$$

where $S\left(\sqrt{1-|x|^2}\right)$ the (one dimensional) sphere with radius $\sqrt{1-|x|^2}$, whose one dimensional geometric measure (length) is $\sqrt{1-|x|^2}\cdot 2\pi$.

Finally, changing to polar coordinates

$$\begin{aligned} 4\pi^2\mathbb{E}\,\mathcal{N}^2 - 4\pi^2\mathcal{D} &= \frac{vol(S^3)(vol(S^1))^2}{\pi^2}\int_0^1 \rho\,\frac{\sigma^4 + |\tau|^2}{(1-\rho^{2d})} d\rho \\ &= 2\,\frac{\pi^2(2\pi)^2}{\pi^2}\,\frac{1}{2}\cdot d^2\frac{d-1}{d} \\ &= 4\pi^2 d(d-1). \end{aligned}$$

Hence $\mathbb{E}\,N^2 = \mathcal{D}^2$ and thus $var(N) = 0$ as claimed.

Case $m > 1$, $d_\ell = 2$

We compute the conditional expectation in Equation (6.9), in this case the structure of the matrices simplify, we have

$$\mathcal{D} = 2^m, \quad \sigma^2 = -\tau = \frac{1-x^2}{1+x^2}; \quad \mathbb{E}\,\zeta_{\ell k}^s\overline{\zeta_{\ell k}^t} = \langle s,t\rangle\,, (k>1)$$

where $x = |\langle s,t\rangle|$

Since the first column is different from the other ones, we expand the determinants in Laplace manner

$$\begin{aligned}\mathbb{E}\,|\det M(s)|^2|\det M(t)|^2 &= \mathbb{E}\left(\det M(s)\overline{M(s)}M(t)\overline{M(t)}\right)\\ &= \sum_{i,j,k,\ell=1}^{m}(-1)^{i+j++k+\ell}\mathbb{E}\left(\zeta_{i1}^s\overline{\zeta_{j1}^s}\zeta_{k1}^t\overline{\zeta_{\ell 1}^t}\right)\cdot\mathbb{E}\left(\det M_{i1}(s)\overline{M_{j1}(s)}M_{k1}(t)\overline{M_{\ell 1}(t)}\right)\end{aligned} \tag{6.10}$$

Note that we can factorize the expectation since each entry only is not independent from the entry in the same position in the other matrix.

Now, using Lemma 6.9, we see that

$$\mathbb{E}\left(\zeta_{i1}^s\overline{\zeta_{j1}^s}\zeta_{k1}^t\overline{\zeta_{\ell 1}^t}\right) = \begin{cases}\sigma^4+|\tau|^2=2\sigma^4, & \text{if } i=j=k=\ell, \text{(case i)}\\ \sigma^2\sigma^2=\sigma^4, & \text{if } i=j\neq k=\ell, \text{(case ii)}\\ |\tau|^2=\sigma^4, & \text{if } i=\ell\neq j=k, \text{(case iii)}\\ 0, & \text{otherwise}\end{cases}$$

In case i, we have

$$\begin{aligned}\mathbb{E}\left(\det M_{i1}(s)\overline{M_{j1}(s)}M_{k1}(t)\overline{M_{\ell 1}(t)}\right) &= \mathbb{E}\left(\det|M_{i1}(s)|^2|M_{i1}(t)|^2\right)\\ &= \mathbb{E}\left(\det|M_{m1}(s)|^2|M_{m1}(t)|^2\right) =: A_{m-1}\end{aligned}$$

That is, we define A_m as the expectation of the determinant of the product of the squared modulus of two $m\times m$ matrices, whose entries are standard Gaussian but each entry is only not independent from the same entry of the other matrix with covariance x.

In case ii, for $i\neq k$

$$\begin{aligned}\mathbb{E}\left(\det M_{i1}(s)\overline{M_{j1}(s)}M_{k1}(t)\overline{M_{\ell 1}(t)}\right) &= \mathbb{E}\left(\det|M_{i1}(s)|^2|M_{k1}(t)|^2\right)\\ &= \mathbb{E}\left(\det|M_{m1}(s)|^2|M_{m-1,1}(t)|^2\right) =: B_{m-1}\end{aligned}$$

That is, in B_m the rows from 1 to $m-1$ have the same law than in A_m, but the last rows of both matrices are independent.

Finally, in case iii, for $i\neq k$:

$$\begin{aligned}\mathbb{E}\left(\det M_{i1}(s)\overline{M_{j1}(s)}M_{k1}(t)\overline{M_{\ell 1}(t)}\right) &= \mathbb{E}\left(\det M_{i1}(s)\overline{M_{i1}(t)M_{k1}(s)}M_{k1}(t)\right)\\ &= \mathbb{E}\left(\det M_{m1}(s)\overline{M_{m1}(t)M_{m-1,1}(s)}M_{m-1,1}(t)\right) =: C_{m-1}\end{aligned}$$

The matrices in C_m have the same law as in B_m; the difference lies in which matrices are conjugated.

Therefore, $(i \neq k = 1, \ldots, m)$,

$$\begin{aligned}\mathbb{E}\left(|\det M(s)|^2 |\det M(t)|^2\right) &= m\,\sigma^4\left(2A_{m-1} + (m-1)\,B_{m-1} + (m-1)\,C_{m-1}\right)\\ &= m(m+1)\,\sigma^4\,A_{m-1}\\ &= (m+1)!(m-1)!\sigma^4\frac{1-x^{2m}}{1-x^2}, \quad (6.11)\end{aligned}$$

where we used Lemma 6.12, see the Auxiliary Results.

Finally, we substitute all this in Formula (6.8). Note that the integrand depends only on x. Therefore:

$$\begin{aligned}&4\pi^2\mathbb{E}\left(N^2\right) - 4\pi^2(2^m)\\ &\quad= (2^m)^2 \cdot vol\left(S^{2m+1}\right) vol\left(S^{2m-1}\right)\frac{(m+1)!(m-1)!}{\pi^{2m}}\\ &\quad= (2\pi)\int_0^1 \frac{(1-x^2)^2}{(1+x^2)^2}\frac{1-x^{2m}}{1-x^2}\frac{1}{(1-x^4)^m}(1-x^2)^{m-1}xdx.\end{aligned}$$

We then have

$$\begin{aligned}4\pi^2\mathbb{E}\left(N^2\right) - 4\pi^2(2^m) &= (2^m)^2\frac{2\pi^{m+1}}{m!}\frac{2\pi^m}{(m-1)!}\frac{(m+1)!(m-1)!}{\pi^{2m}}\\ &\cdot (2\pi)\int_0^1 \frac{(1-x^2)^2}{(1+x^2)^2}\frac{1-x^{2m}}{1-x^2}\frac{1}{(1-x^4)^m}(1-x^2)^{m-1}(xdx)\\ &= 4\pi^2\cdot(2^m)^2(m+1)\int_0^1\frac{1-x^{2m}}{(1+x^2)^{m+2}}(2xdx)\end{aligned}$$

Making the change of variables $1+x^2 \mapsto u$ in the integral we obtain:

$$\begin{aligned}\int_0^1 \frac{x^{2m}}{(1+x^2)^{m+2}}(2xdx) &= \int_1^2 \frac{(u-1)^m}{u^{m+2}}du\\ &= \int_1^2 \left(1-\frac{1}{u}\right)^m\frac{1}{u^2}du = \frac{1}{m+1}\left(1-\frac{1}{u}\right)^{m+1}\Bigg|_1^2\\ &= \frac{1}{m+1}\frac{1}{2^{m+1}}\end{aligned}$$

Thus

$$\int_0^1 \frac{1-x^{2m}}{(1+x^2)^{m+2}}(2xdx) = \frac{1}{m+1}\left[1-\frac{1}{2^{m+1}}\right] - \frac{1}{m+1}\frac{1}{2^{m+1}} = \frac{2^m-1}{2^m(m+1)}$$

$$\mathbb{E}\, N^2 = (2^m)\,[1 + (2^m - 1)] = (2^m)^2$$

as claimed. Thus $var(N) = 0$.

6.4.4 Equal degrees, $d_\ell = d$ for all ℓ

The same arguments as in the quadratic case ($d = 2$) should allow us to state the case of equal degrees.

The difference in the determinant is that the quantities σ^2 and $-\tau$ do not coincide anymore. Therefore, with the same notations as above, the determinant is

$$\mathbb{E}\,|\det M(s)|^2 |\det M(t)|^2 = \sigma^4 B_m + \frac{|\tau|^2}{x^{2d-2}} C_m$$

where $x = |\,\langle s, t\rangle\,|$, and on the factors B_m and C_m the x is replaced by x^{d-1}.

Unfortunately, this gets very involved.

6.5 Auxiliary computations

For the sake of readability, we write down here some computations needed in the previous sections.

6.5.1 About the Gaussian distribution

Lemma 6.9. *Let (S, T) be jointly centered, complex Gaussian random variables with variance 1 and covariance ρ. Denote S_r, T_r and S_{im}, T_{im} for the real and imaginary parts of S and T respectively, denote $\rho_{i,j} = \mathbb{E}\,(S_i T_j)$ for $i, j = r, im$. Then*

$$\begin{aligned} \rho_{r,r} &= \rho_{im,im} = \frac{1}{2}\mathbb{R}e(\rho) \\ \rho_{r,im} &= -\rho_{im,r} = -\frac{1}{2} Im(\rho) \end{aligned}$$

This follows directly from the definition.

Lemma 6.10. *Let (S, T) be jointly centered, Gaussian random variables with variance 1 and covariance ρ. Then*

1. *on the real case* $\mathbb{E}\left(|S|^2|T|^2\right) = 1 + 2\rho^2$.

2. *on the complex case* $\mathbb{E}\left(|S|^2|T|^2\right) = 1 + |\rho|^2$.

Proof. Real case: Let S, W be two real independent, centered, Gaussian random variables and write $T = \rho S + \sqrt{1-\rho^2}W$, then

$$\mathbb{E}\left(S^2T^2\right) = \rho^2\mathbb{E}\,S^4 + 2\rho\sqrt{1-\rho^2}\mathbb{E}\,SW + (1-\rho^2)\mathbb{E}\,W^2 = 1 + 2\rho^2.$$

Complex case: Use the real case for the real and imaginary parts, taking into account that these random variables have half of the variance than the real ones. □

6.5.2 Computation of the covariances of the derivatives

Fix $s, t \in \mathbb{C}^{m+1}$. Let $\{v_2, \ldots, v_m\}$ be an orthonormal set in $\mathbb{C}^{m+1}$ such that $v_k \perp s, t$, $(k \geq 2)$. Define

$$v_s = \frac{t - \langle t, s\rangle\, s}{\sqrt{1 - |\langle s, t\rangle|^2}}, \qquad v_t = \frac{s - \langle s, t\rangle\, t}{\sqrt{1 - |\langle s, t\rangle|^2}}.$$

Then $B_s := \{v_s, v_2, \ldots, v_m\}$ and $B_t := \{v_t, v_2, \ldots, v_m\}$ are orthonormal basis of T_sS^m and T_tS^m respectively.

It is easy to see that

$$\langle s, v_t\rangle = \langle t, v_s\rangle = \sqrt{1 - |\langle s, t\rangle|^2}, \qquad \langle v_s, v_t\rangle = -\langle t, s\rangle.$$

Denote $\partial_k f(w)$ for $\frac{\partial f}{\partial v_k}(w)$, $k = s, t, 2, \ldots, m$. and express all the derivatives on these basis, that is

$$f'(s) = \begin{pmatrix} \partial_s f_1(s) & \partial_2 f_1(s) & \ldots & \partial_m f_1(s) \\ \partial_s f_2(s) & \partial_2 f_2(s) & \ldots & \partial_m f_2(s) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_s f_m(s) & \partial_2 f_m(s) & \ldots & \partial_m f_(s) \end{pmatrix},$$

$$f'(t) = \begin{pmatrix} \partial_t f_1(t) & \partial_2 f_1(t) & \ldots & \partial_m f_1(t) \\ \partial_t f_2(t) & \partial_2 f_2(t) & \ldots & \partial_m f_2(t) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_t f_m(t) & \partial_2 f_m(t) & \ldots & \partial_m f_(t) \end{pmatrix}$$

Lemma 6.11. *For the sake of simplicity, we omit the sub index ℓ. Let $s, t \in \mathbb{C}^{m+1}$ be such that $\|s\| = \|t\| = 1$, then*

	$k = s$	$k = t$	$k \geq 2$
$\mathbb{E}\, \partial_k f_\ell(t)\overline{f_\ell(t)}$	0		0
$\mathbb{E}\, \partial_k f_\ell(t)\overline{f_\ell(s)}$	$d_\ell \langle t, s\rangle^{d_\ell - 1} \sqrt{1 - \lvert \langle s, t\rangle \rvert^2}$		0
$\mathbb{E}\, \partial_k f_\ell(s)\overline{f_\ell(s)}$		0	0
$\mathbb{E}\, \partial_k f_\ell(s)\overline{f_\ell(t)}$		$d_\ell \langle s, t\rangle^{d_\ell - 1} \sqrt{1 - \lvert \langle s, t\rangle \rvert^2}$	0

Furthermore

$$\mathbb{E}\, \partial_s f_\ell(s)\overline{\partial_t f_\ell(t)} = d_\ell(d_\ell - 1) \langle s, t\rangle^{d_\ell - 2} (1 - \lvert \langle s, t\rangle \rvert^2) - d_\ell \langle s, t\rangle_\ell^d$$

Proof. We start with the first row. Since $\mathbb{E}\, f(s)\overline{f(s)} \equiv 1$ on the sphere and we take derivatives on the tangent space, these derivatives (w.r.t. $v_s, v_2, \ldots, v_m$) vanish. The third row is analogous.

Let us prove now the second row, the fourth one is analogous. Since $\langle t, v_k\rangle = 0$ for $k \geq 2$, we have that $\mathbb{E}\, \partial_k f_\ell(t)\overline{f_\ell(s)} = 0$. Besides

$$\begin{aligned} \mathbb{E}\, f(s)\overline{\partial_t f(t)} &= \overline{\partial_t \langle t, s\rangle^d} \\ &= \overline{d \langle t, s\rangle^{d-1} \frac{\partial}{\partial v_t} \langle t, s\rangle} = d \langle s, t\rangle^{d-1} \langle s, v_t\rangle \\ &= d \langle s, t\rangle^{d-1} \sqrt{1 - \lvert \langle s, t\rangle \rvert^2}. \end{aligned}$$

Now, we move to the final formula. Taking derivative with respect to v_s we have

$$\begin{aligned} \mathbb{E}\, \partial_s f(s)\overline{\partial_t f(t)} &= \frac{\partial}{\partial v_s} \left(d \langle s, t\rangle^{d-1} \langle s, v_t\rangle \right) \\ &= d(d-1) \langle s, t\rangle^{d-2} (1 - \lvert \langle s, t\rangle \rvert^2) - d \langle s, t\rangle^d. \end{aligned}$$

This concludes the proof. □

6.5.3 Computation of τ and σ^2

By definition, after Equation (6.9), τ equals

$$\tau = \mathbb{E}\, (\partial_s f(s) - \alpha_s f(s) - \beta_s f(t)) \overline{\partial_t f(t)}$$

Then, by Lemma 6.11

$$\begin{aligned}
\tau &= d(d-1)\langle s,t\rangle^{d-2}(1-|\langle s,t\rangle|^2) - d\langle s,t\rangle^{d-2}|\langle s,t\rangle|^2 \\
&\quad + d^2\langle s,t\rangle^{d-2}|\langle s,t\rangle|^{2d}\frac{1-|\langle s,t\rangle|^2}{1-|\langle s,t\rangle|^{2d}} \\
&= d\langle s,t\rangle^{d-2}\left[(d-1)(1-|\langle s,t\rangle|^2) - |\langle s,t\rangle|^2 + d|\langle s,t\rangle|^{2d}\frac{1-|\langle s,t\rangle|^2}{1-|\langle s,t\rangle|^{2d}}\right] \\
&= d\langle s,t\rangle^{d-2}\left[-1 + d(1-|\langle s,t\rangle|^2)\left(1+\frac{|\langle s,t\rangle|^{2d}}{1-|\langle s,t\rangle|^{2d}}\right)\right] \\
&= d\langle s,t\rangle^{d-2}\left[-1 + d\frac{1-|\langle s,t\rangle|^2}{1-|\langle s,t\rangle|^{2d}}\right]
\end{aligned}$$

On the other hand, also by its definition

$$\sigma^2 = \mathbb{E}\left(\partial_s f(s) - \alpha_s f(s) - \beta_s f(t)\right)\overline{\partial_s f(s)}$$

Again by Lemma 6.11

$$\sigma^2 = d\left[1 - d|\langle s,t\rangle|^{2d-2}\frac{(1-|\langle s,t\rangle|^2)}{1-|\langle s,t\rangle|^{2d}}\right]$$

6.5.4 Recurrence for the expectation of the determinant

We present here for the sake of readability, as an auxiliary result the recurrence for the computation of the expectation of the determinant.

Lemma 6.12. *We have* $A_m = B_m + C_m$ *and*

$$A_m = (m!)^2\frac{1-x^{2m}}{1-x^2}.$$

Proof. We expand the determinants in Laplace manner in order to obtain recurrence equations for the quantities A_m, B_m and C_m. For simplicity denote $M = (a_{ij})$, $N = (b_{ij})$ with a_{ij} and (b_{ij}) i.i.d. standard Gaussian random variables such that $cov(a_{ij}, b_{ij}) = x$, the rest being independent, i.e: $cov(a_{ij}, b_{i'j'}) = 0$ for $(i,j) \neq (i',j')$.

Then

$$A_m = \sum_{i,j,k,\ell=1}^{m}(-1)^{i+j+k+\ell}\mathbb{E}\, a_{mi}\overline{a_{mj}}\,\overline{b_{mk}}b_{m\ell}\,\mathbb{E}\det M_{mi}\overline{M_{mj}N_{mk}}N_{m\ell}$$

Now

$$\mathbb{E}\, a_{mi}\overline{a_{mj}}\,\overline{b_{mk}}b_{m\ell} = \begin{cases} 1+x^2, & \text{if } i=j=k=\ell, \text{(case i)} \\ 1, & \text{if } i=j\neq k=\ell, \text{(case ii)} \\ x^2, & \text{if } i=\ell\neq j=k, \text{(case iii)} \\ 0, & \text{otherwise} \end{cases}$$

Hence

$$\begin{aligned} A_m &= (1+x^2)\sum_{i=1}^{m}\mathbb{E}\,|\det M_{mi}|^2|\det N_{mi}|^2 \\ &\quad + \sum_{i\neq k=1}^{m}\mathbb{E}\,|\det M_{mi}|^2|\det N_{mk}|^2 + x^2\sum_{i\neq k=1}^{m}\mathbb{E}\,\det M_{mi}\overline{M_{mk}N_{mk}}N_{mi} \\ &= m(1+x^2)\,A_{m-1} + m(m-1)\,B_{m-1} + m(m-1)x^2\,B_{m-1} \end{aligned}$$

For B_m we assume that the last row in M is independent from the last row in N, then:

$$B_m = \sum_{i,j,k,\ell=1}^{m}(-1)^{i+j+k+\ell}\mathbb{E}\, a_{mi}\overline{a_{mj}}\,\overline{b_{mk}}b_{m\ell}\,\mathbb{E}\,\det M_{mi}\overline{M_{mj}N_{mk}}N_{m\ell}$$

Now

$$\begin{aligned} \mathbb{E}\, a_{mi}\overline{a_{mj}}\,\overline{b_{mk}}b_{m\ell} &= \mathbb{E}\, a_{mi}\overline{a_{mj}}\,\mathbb{E}\,\overline{b_{mk}}b_{m\ell} \\ &= \begin{cases} 1, & \text{if } i=j \text{ and } k=\ell \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Then

$$\begin{aligned} B_m &= \sum_{i,k=1}^{m}\mathbb{E}\,|\det M_{mi}|^2|\det N_{mk}|^2 \\ &= \sum_{i=1}^{m}\mathbb{E}\,|\det M_{mi}|^2|\det N_{mi}|^2 + \sum_{i\neq k=1}^{m}\mathbb{E}\,|\det M_{mi}|^2|\det N_{mk}|^2 \\ &= m\mathbb{E}\,|\det M_{m1}|^2|\det N_{m1}|^2 + m(m-1)\mathbb{E}\,|\det M_{m1}|^2|\det N_{m2}|^2 \\ &= mA_{m-1} + m(m-1)B_{m-1}. \end{aligned}$$

With the same assumptions as in the case of B_m, we expand the determinant in Laplace manner along row $m-1$ for the first two factors and along row m for

the remaining two. Then

$$\begin{aligned}C_{m-1} &= \mathbb{E}\det M_{m1}(s)\overline{M_{m1}(t)M_{m-1,1}(s)}M_{m-1,1}(t)\\ &= \sum_{i,j,k,\ell=2}^{m}(-1)^{i+j+k+\ell}\mathbb{E}\, a_{m-1,i}\overline{b_{m-1,j}a_{mk}}b_{m\ell}\\ &\quad\cdot \mathbb{E}\,\det M(s)_{\{m-1,m\}\times\{1,i\}}\overline{M(t)_{\{m-1,m\}\times\{1,j\}}}\\ &\quad\cdot \det\overline{M(s)_{\{m-1,m\}\times\{1,k\}}}M(t)_{\{m-1,m\}\times\{1,\ell\}}\end{aligned}$$

Now

$$\begin{aligned}\mathbb{E}\, a_{m-1,i}\overline{b_{m-1,j}a_{mk}}b_{m\ell} &= \mathbb{E}\, a_{m-1,i}\overline{b_{m-1,j}}\mathbb{E}\, a_{mk}b_{m\ell}\\ &= \begin{cases}x^2, & \text{if } i=j \text{ and } k=\ell\\ 0, & \text{otherwise}\end{cases}\end{aligned}$$

Therefore

$$\begin{aligned}C_{m-1} &= x^2\sum_{i,k=2}^{m}\mathbb{E}\det M(s)_{\{m-1,m\}\times\{1,i\}}\overline{M(t)_{\{m-1,m\}\times\{1,i\}}}\\ &\quad\cdot\mathbb{E}\det\overline{M(s)_{\{m-1,m\}\times\{1,k\}}}M(t)_{\{m-1,m\}\times\{1,k\}}\\ &= x^2\sum_{i=2}^{m}\mathbb{E}\,|\det M_{\{m-1,m\}\times\{1,i\}}(s)|^2|\det M_{\{m-1,m\}\times\{1,i\}}(t)|^2\\ &\quad + x^2\sum_{i\neq k=2}^{m}\mathbb{E}\det M(s)_{\{m-1,m\}\times\{1,i\}}\overline{M(t)_{\{m-1,m\}\times\{1,i\}}}\\ &\quad\cdot\mathbb{E}\det\overline{M(s)_{\{m-1,m\}\times\{1,k\}}}M(t)_{\{m-1,m\}\times\{1,k\}}\end{aligned}$$

Since, all the terms in each sum involve matrices with the same distribution

$$\begin{aligned}C_{m-1} &= (m-1)x^2\mathbb{E}\,|\det M_{\{m-1,m\}\times\{1,2\}}(s)|^2|\det M_{\{m-1,m\}\times\{1,2\}}(t)|^2\\ &\quad + (m-1)(m-2)x^2\mathbb{E}\det M(s)_{\{m-1,m\}\times\{1,i\}}\overline{M(t)_{\{m-1,m\}\times\{1,i\}}}\\ &\quad\cdot\mathbb{E}\det\overline{M(s)_{\{m-1,m\}\times\{1,k\}}}M(t)_{\{m-1,m\}\times\{1,k\}}\\ &= (m-1)x^2A_{m-2}+(m-1)(m-2)x^2C_{m-2}\end{aligned}$$

Hence

$$C_m = mx^2A_{m-1}+m(m-1)x^2C_{m-1}$$

Therefore, we have the system

$$\begin{pmatrix}A_m\\B_m\\C_m\end{pmatrix} = \begin{pmatrix}m(1+x^2) & m(m-1) & m(m-1)x^2\\ m & m(m-1) & 0\\ mx^2 & 0 & m(m-1)x^2\end{pmatrix}\begin{pmatrix}A_{m-1}\\B_{m-1}\\C_{m-1}\end{pmatrix}$$

It follows that $A_m = B_m + C_m$ and that the recurrence for the pair A_m, B_m is

$$\begin{pmatrix} A_m \\ B_m \end{pmatrix} = \begin{pmatrix} m(1+mx^2) & m(m-1)(1-x^2) \\ m & m(m-1) \end{pmatrix} \begin{pmatrix} A_{m-1} \\ B_{m-1} \end{pmatrix} \tag{6.12}$$

The solution for this system is:

$$A_m = (m!)^2 \frac{1-x^{2m}}{1-x^2}$$
$$B_m = \frac{m!(m-1)!}{1-x^2}\left(m - \frac{1-x^{2m+2}}{1-x^2}\right)$$

This claim can be directly checked. □

6.5.5 Alternative proof for the expectation

We include this section for comparison of the real and complex frameworks.

Lemma 6.13. *The random variables*

$$\frac{\partial u_\ell}{\partial x_h}(e_0,0), \frac{\partial u_\ell}{\partial y_k}(e_0,0), \frac{\partial v_\ell}{\partial x_h}(e_0,0) \text{ and } \frac{\partial v_\ell}{\partial x_k}(e_0,0)$$

with $1 \leq h \leq m, 0 \leq k \leq m$ *are independent centered Gaussian with variances* $d_\ell/2$.

Proof. From $\mathbb{E}\,(u_\ell(t))^2 = \mathbb{E}\,(v_\ell(t))^2 = \|t\|^2/2$, taking derivatives it follows the claimed independence and variances.

The rows of $Df(e_0,0)$ corresponding to $u_\ell(e_0,0)$ and $v_\ell(e_0,0)$ are vectors of independent centered Gaussian variables with variance $d_\ell/2$, □

It follows that

$$\mathbb{E}\,|\det(Df(t_0)\cdot Df(t_0)')|^{1/2} = \frac{\mathcal{D}}{2^m}\mathbb{E}\,|\det(G_{2m,2m+1}\cdot G'_{2m,2m+1}))|^{1/2}.$$

where $\mathcal{D}$ is Bézout number and G is a $(2m)\times(2m+1)$ matrix with i.i.d. standard Gaussian entries.

The modulus of the determinant is the square of the volume generated by the $2m$ rows of G in $\mathbb{R}^{2m+1}$, hence we can apply the computations in A¡aïs & Wschebor [12, page 305] with $2m+1$ instead of m and starting at $k=2$ since

when projecting the first vector the co-dimension of the linear space spanned by the remaining rows is 2:

$$\begin{aligned}\mathbb{E}\,|\det(G_{2m,2m+1}\cdot G'_{2m,2m+1}))|^{1/2} &= \mathbb{E}\,|vol_{2m}(G)^2|^{1/2}\\ &= \prod_{k=2}^{2m+1}\mathbb{E}\,\|\eta_k\|_{(k)}\end{aligned}$$

where $\eta_k \sim N(0, Id_k)$. elementary computations show that

$$\begin{aligned}\prod_{k=2}^{2m+1}\mathbb{E}\,\|\eta_k\|_{(k)} &= \prod_{k=2}^{2m+1}\left[\sqrt{2}\frac{\Gamma((k+1)/2)}{\Gamma(k/2)}\right]\\ &= 2^m\Gamma\,(m+1)\end{aligned}$$

In conclusion:

$$\begin{aligned}\mathbb{E}\,N &= \frac{1}{2\pi}\frac{2\pi^{m+1}}{\Gamma(m+1)}\cdot\frac{1}{\pi^m}\cdot\mathbb{E}\,|\det(Df(t_0)\cdot Df(t_0)')|^{1/2}\\ &= \frac{1}{2\pi}\frac{2\pi^{m+1}}{\Gamma(m+1)}\cdot\frac{1}{\pi^m}\cdot\frac{\mathcal{D}}{2^m}\cdot\mathbb{E}\,|\det(G_{2m,2m+1}\cdot G'_{2m,2m+1}))|^{1/2}\\ &= \frac{1}{2\pi}\frac{2\pi^{m+1}}{\Gamma(m+1)}\cdot\frac{1}{\pi^m}\cdot\frac{\mathcal{D}}{2^m}\cdot 2^m\Gamma(m+1)\\ &=\\ D.\end{aligned}$$

This concludes the proof.

Appendices

Appendix A: Gaussian Processes

Definition

Let $(\Omega, \mathcal{M}, \mathbb{P})$ be a probability space, we say that a random vector $X : \Omega \to \mathbb{R}^d$ has the normal, or Gaussian, distribution if its distribution $\mathbb{P}_X := \mathbb{P} \circ X^{-1} : \mathcal{B}_d \to \mathbb{R}$, where $\mathcal{B}_d$ is Borel sigma - algebra, has the density function, with respect to Lebesgue measure given by:

$$p(x) = \frac{1}{(2\pi)^{d/2}\sqrt{\det(\Sigma)}} \exp\left[-\frac{1}{2}(x-\mu)^t \Sigma^{-1}(x-\mu)\right],$$

where $\mu \in \mathbb{R}^d$ is the mean of X, and $\Sigma = (\sigma_{ij})$ is a $d \times d$ non-negative definite matrix, called the matrix of variances of X. When $\mu = 0$ and $\Sigma = Id$ we say that X is a standard Gaussian vector.

It is easy to check that $\mathbb{E}\, X = \mu$ and $cov(X_i, X_j) = \sigma_{ij}$. It is possible to extend the definition to the degenerated case, that is when Σ is singular, using the characteristic function, but we will not need such an extension on this work.

It follows from the definition that the Gaussian distribution depends only on two parameters, namely the mean μ and the variance matrix Σ. Thus, we say that this law is of second order.

This distribution has very peculiar properties that make it special, we mention some of them needed in the thesis below.

Gaussian regression

Let (X, Y) be a Gaussian vector in $\mathbb{R}^{d+d'}$ with mean (μ_X, μ_Y), then, in order to show that X and Y are independent it suffices to show that their covariance vanish, that is, $cov(X, Y) = \mathbb{E}\left[(X - \mu_X)(Y - \mu_Y)\right] = 0$. On the other hand, any linear (affine) combination of Gaussian vectors is again a Gaussian vector whose parameters are linear (affine) combinations of those of (X, Y).

One fundamental consequence of these properties is that they permit to define a natural version of the conditional expectation of X w.r.t. Y. At this point it is worth to say that in the general situation the computation of the conditional expectation is a very demanding problem.

Let $f : \mathbb{R}^d \to \mathbb{R}$ a bounded function, then

$$\mathbb{E}\left[f(X) \mid Y = y\right] = \mathbb{E}\, f(\zeta + Cy),$$

where ζ is a Gaussian random variable with mean $\mu_X - C\mu_Y$ and variance $var(X) + C \cdot cov(X, Y)$, being $C = cov(X, Y)var(Y)^{-1}$.

In general, the idea is to choose the deterministic matrix C such that the vectors be non-correlated, and thus independent.

Continuity of moments and paths

We say that an stochastic process (or field) $X : (\Omega\times)T \to \mathbb{R}^d$ is a Gaussian process (or field) if the finite-dimensional distributions are Gaussian. Kolmogorov Extension Theorem show that the distribution of the process is determined by order one and order two moments, that is, by the mean function $\mu : T \to \mathbb{R}^d$ and the covariance function $r : T \times T \to \mathbb{R}^{d\times d}$.

The paths of the process are the functions obtained by fixing $\omega \in \Omega$, that is, the paths of $\mathcal{X}$ are the functions $X(\omega, \cdot) : T \to \mathbb{R}^d$ for $\omega \in \Omega$.

It is a remarkable fact that on the Gaussian case the regularity of these functions (mean, covariance and paths) are intimately related, see Azaïs & Wschebor [12, Chapter 2] or Cramér & Leadbetter [23] for detailed treatment. Roughly speaking, to C^k paths correspond C^{2k} covariance function and vice versa.

Spectral representation

We say that the centered Gaussian process X is stationary if the covariance function $r(s, t)$ depends on s, t only through their difference $t - s$.

On this case, since r is positive definite, by Bochner's Theorem, it can be represented as

$$r(\tau) = \int_{\mathbb{R}} e^{i\tau x}\mu(dx),$$

for some Borel measure μ, called the spectral measure of X.

The second spectral moment of X at $t = 0$ is defined as

$$\lambda_2 = \lambda_2(X(0)) = \int_{\mathbb{R}} \lambda^2 \mu(d\lambda).$$

It is well known from Spectral Analysis that λ_2 measures, to some extent, the roughness of the paths of X, that is, to larger values of λ_2 more cusps appear in the paths. In other words, as the value of λ_2 increases the paths have more components of higher frequencies, see Adler & Taylor [1, Chapter 11].

Besides, if the process centered stationary Gaussian process X has paths of class C^1, it follows that

$$\lambda_2 = -r''(0).$$

In words, the second spectral moment coincides with the variance of the process X' at $t = 0$.

Invariance

When the dimension of the parameter space T is larger than one, the property of stationarity is replaced by that of being invariant under the action of the orthogonal (or unitary) group of transformations of T.

In practice, this condition deals to important simplifications in the integrals, conditional expectations, etc.

Appendix B: Point Processes

In order to describe the jump structure of an stochastic process it is convenient to introduce the so called Point Processes. in the following lines we write down some of their main properties, we follow the book by Jacobsen [42], another very well known reference for the subject is the book by Jacod & Shiryaev [43].

A marked point process (MPP for short) is a sequence $(\tau_n, \xi_n)_{n\in\mathbb{N}}$, where $\tau_n < \tau_{n+1}$ for all n. Here $\tau_n \in [0,\infty], n \in \mathbb{N}$, represents the n-the jump epoch of the process and $\xi_n \in E, n \in \mathbb{N}$, with $(E, \mathcal{E})$, represents the mark corresponding to the jump epoch τ_n. Here E is $\mathbb{R}$ (or $\mathbb{R}^2$) and $\mathcal{E}$ the Borel σ-algebra.

This kind of stochastic processes model the occurrence through time of events of different nature. The main example of a MPP is the Compound Poisson Process (CPP for short), on this case the intervals between two consecutive jump epochs (inter arrival intervals) are independent and with identically distributed exponential length; besides, the marks are also independent identically distributed random variables independent from the jump epochs.

Construction

In order to construct such a process, we need two families of stochastic Markov kernels $P^{(n)}_{x,t}$ and $\pi^{(n)}_{x,t}$, with $n \in \mathbb{N}$, $x \in E^n$ y $t \in [0,\infty]^n$. The kernels specify the (conditional) distributions of the jump epochs and marks, conditioned to the previous times and marks.

For notational convenience, sometimes, we will write only the relevant components of the conditions.

set $\tau_0 = 0$ and draw ξ_0 according to the initial distribution π_0, then, conditioned on the resulting value of ξ_0, say x_0, draw τ_1 with (conditional) distribution

$$\mathbb{P}\left(\tau_1 \in \cdot \mid \xi_0 = x_0\right) = P^{(1)}_{x_0}(\cdot).$$

Similarly, conditioned on the values of ξ_0, τ_1, say x_0, t_1 respectively, draw ξ_1 with

distribution $\pi^{(1)}_{x_0,t_1}(\cdot)$. That is

$$\mathbb{P}\left(\xi_1 \in \cdot \mid \xi_0 = x_0, \tau_1 = t_1\right) = \pi^{(1)}_{x_0,t_1}(\cdot).$$

Then, conditioned on the preceding values and on $\xi_1 = x_1$ draw τ_2 with distribution $P^{(2)}_{(x_0,x_1),t_1}(\cdot)$ and so on.

Finally, for $\tau_n \leq t < \tau_{n+1}$ let $\nu_t = n$ and

$$J(t) = \sum_{k=0}^{\nu_t} \xi_k.$$

Hence, τ_n represents the n-th jump instant and ξ_n represents the n-th jump magnitude (or increment) of the process $\mathcal{J}$.

Counting Processes and Random Counting Measures

Departing from the point process $(\tau_n, \xi_n)_n$ we define the following counting processes

$$\begin{aligned} \nu_t &= \max\{n : \tau_n \leq t\} = \#\{n : \tau_n \leq t\} \\ \nu_t(A) &= \#\{n : \tau_n \leq t, \xi_n \in A\}, A \in \mathcal{E}. \end{aligned}$$

Therefore, ν_t represents the number of jumps epochs occurring before t, that is the jump epochs on the interval $[0, t]$, and $\nu_t(A)$ represents the number of these jump epochs whose marks belong to the set A.

It is easy to see that $\{\nu_t \geq n\} = \{\tau_n \leq t\}$ and $\tau_n = \inf\{t : \nu_t \leq n\}$.

A point process defines also a random counting measure (RCM) on $[0, \infty] \times E$ by

$$\mu = \sum_n \delta_{(\tau_n,\xi_n)},$$

where δ_x is Dirac Delta measure concentrated at the point x.

The basic properties of the RCM are

$$\begin{aligned} \mu(\{0\} \times E) &= 0 \\ \mu(\{t\} \times E) &\leq 1 \\ \mu((0, t] \times E) &< \infty, \end{aligned} \tag{6.13}$$

for all t.

Adapted and predictables processes

Let $\mathcal{M}$ be the space of counting measures satisfying the conditions in Equations (6.13), in $\mathcal{M}$ define the filtration

$$\mathcal{H}_t = \sigma\left(\nu_s(A) : 0 \leq s \leq t, A \in \mathcal{E}\right).$$

In this framework, an stochastic process is a function $X : [0, \infty] \times \mathcal{M} \to \mathbb{R}$ measurable w.r.t. $\mathcal{B} \otimes \mathcal{H}_\infty$. besides, an adapted process $\mathcal{X}$ is a process such that $X(t)$ is $\mathcal{H}_t$-measurable and an predictable process $\mathcal{X}$ is a process such that $X(t)$ is $\mathcal{H}_{t^-}$-measurable.

Roughly speaking, a process is adapted if its value at time t depend upon the observed jump epochs and marks occurred until time t inclusive; while a process is predictable if its value at time t can be predicted from the observed jump epochs and marks occurring in $[0, t)$, but not those occurring at time t.

Compensators

The counting processes $(\nu_t)_t$ and $\nu_t(A)$ are non-decreasing adapted processes, therefore, they are sub martingales. One of the fundamental consequences of this fact is Doob-Meyer decomposition, see Jacobsen [42, Th. 4.6.1]. This result states that these counting processes can be represented as the sum of a (local) martingale and an increasing predictable process.

The predictable process appearing in this decomposition is called the compensator of the counting process, see Jacobsen [42, Def. 4..3.2].

Integrals and Martingales

As the RCM and the compensating measures are measures, we can integrate with respect to them.

It is a fundamental result that the difference of these integrals is a martingale under very mild conditions.

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